

# Light composite scalar and other spectra in $N_f=8$ QCD



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Lattice2014, June 27, Columbia Univ.

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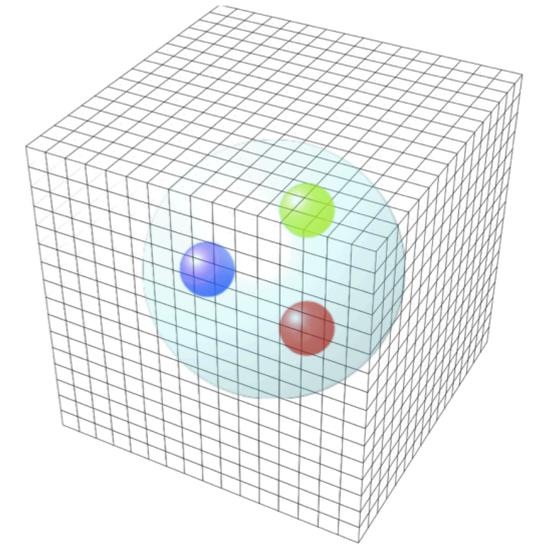
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# Plan of Talk:



## 1. Introduction

## 2. Walking signal of $N_f=8$ LQCD

{ Chiral Perturbation Theory (ChPT)  
{ Finite Size Hyperscaling (FSHS)

♠  $N_f=8$  is a candidate for the walking.

Walking signals in  $N_f=8$  QCD on the lattice

Published in *Phys.Rev. D87 (2013) 094511*, e-Print: [arXiv:1302.6859 \[hep-lat\]](https://arxiv.org/abs/1302.6859).

+ update

## 3. Light composite scalar in $N_f=8$

♠ Light scalar in  $N_f=8$  .

Light composite scalar in eight-flavor QCD on the lattice

Published in *Phys.Rev. D89, 111502(R) (2014)*, e-Print: [arXiv:1403.5000 \[hep-lat\]](https://arxiv.org/abs/1403.5000).

## 4. Summary

# 1. Introduction

# Walking technicolor

$N_f$  massless fermions +  $SU(N_{TC})$  gauge at  $O(1)$  TeV

Model requirement:

- Spontaneous chiral symmetry breaking
- Slow running (walking) coupling in wide scale range
- Large anomalous mass dimension  $\gamma^* \sim 1$  in walking region

- Higgs  $\approx$  Light composite scalar  
pNGB (technidilaton)  
of scale symmetry breaking



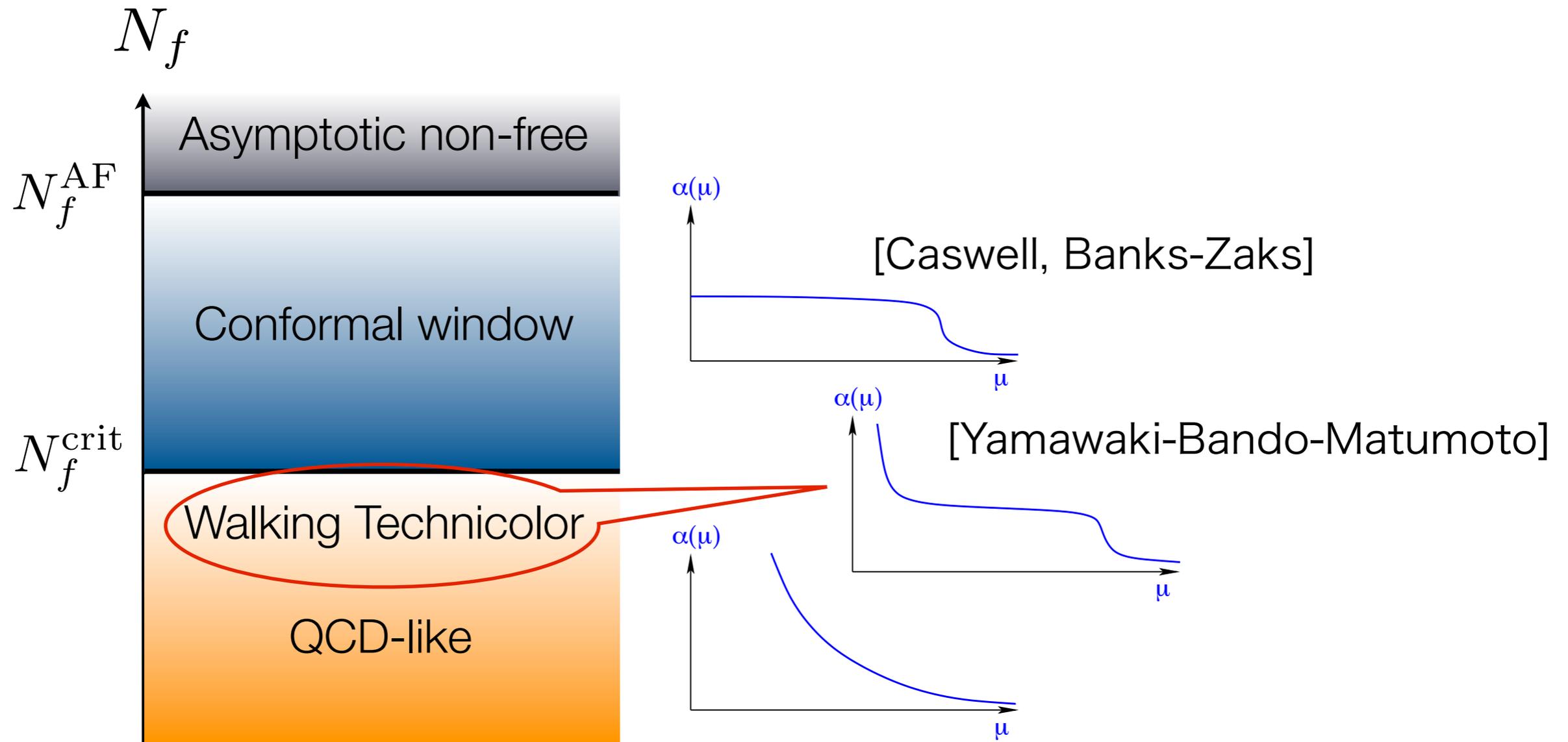
$$m_{\text{Higgs}}/v_{\text{EW}} \sim 0.5 = m_{\sigma}/(\sqrt{N_d}F)$$

$F$  : decay constant,  $N_d$  : number of weak doublets

usual QCD  $m_{\sigma}/F \sim 4-5$

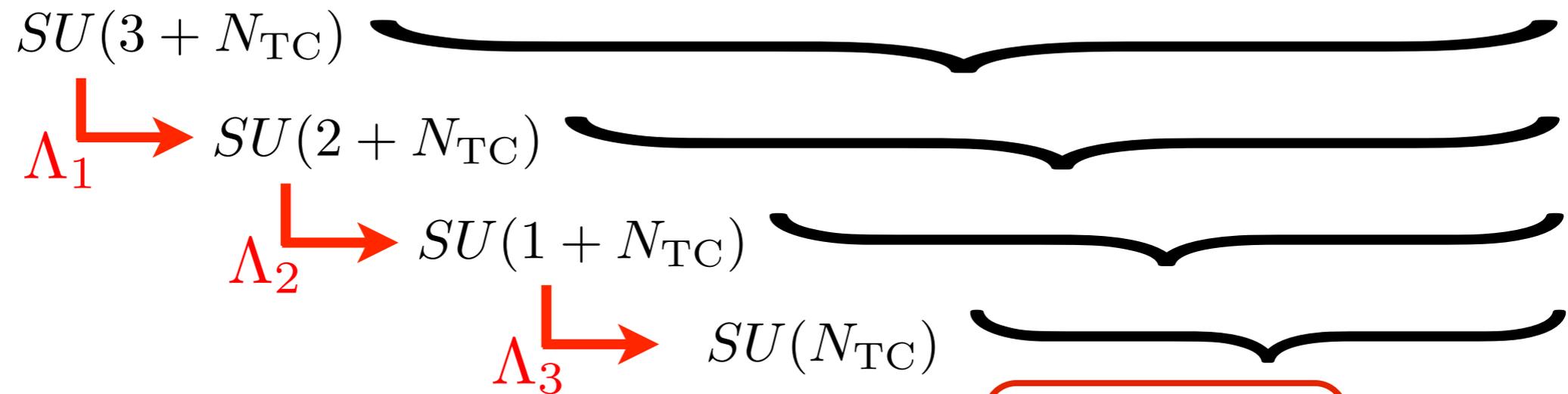
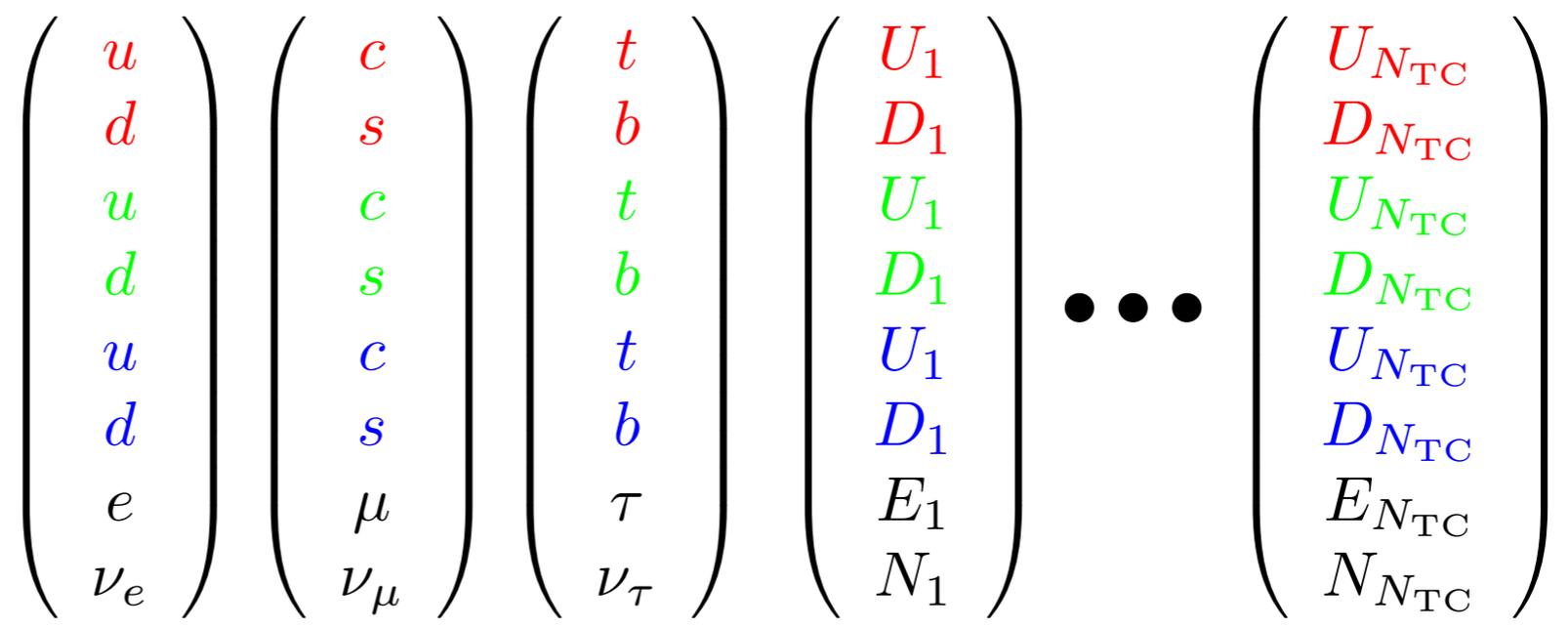
# conformal window and walking coupling

- non-Abelian gauge theory with  $N_f$  massless fermions -



- Walking Technicolor could be realized just below the conformal window
- crucial information:  $N_f^{crit}$  & mass anomalous dimension around  $N_f^{crit}$

One-family  
Extended  
TechniColor  
model



8-flavor  $SU(N_{TC})$   
technicolor

Why  $N_f=8$  QCD?

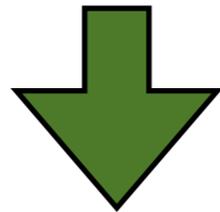
We consider  
 $N_{TC} = 3$

Many flavor QCD  
⇒ Candidate for walking/conformal

Our investigation in  $N_f=12$   
→ consistent with the conformal with  $\gamma = 0.4--0.5$ .

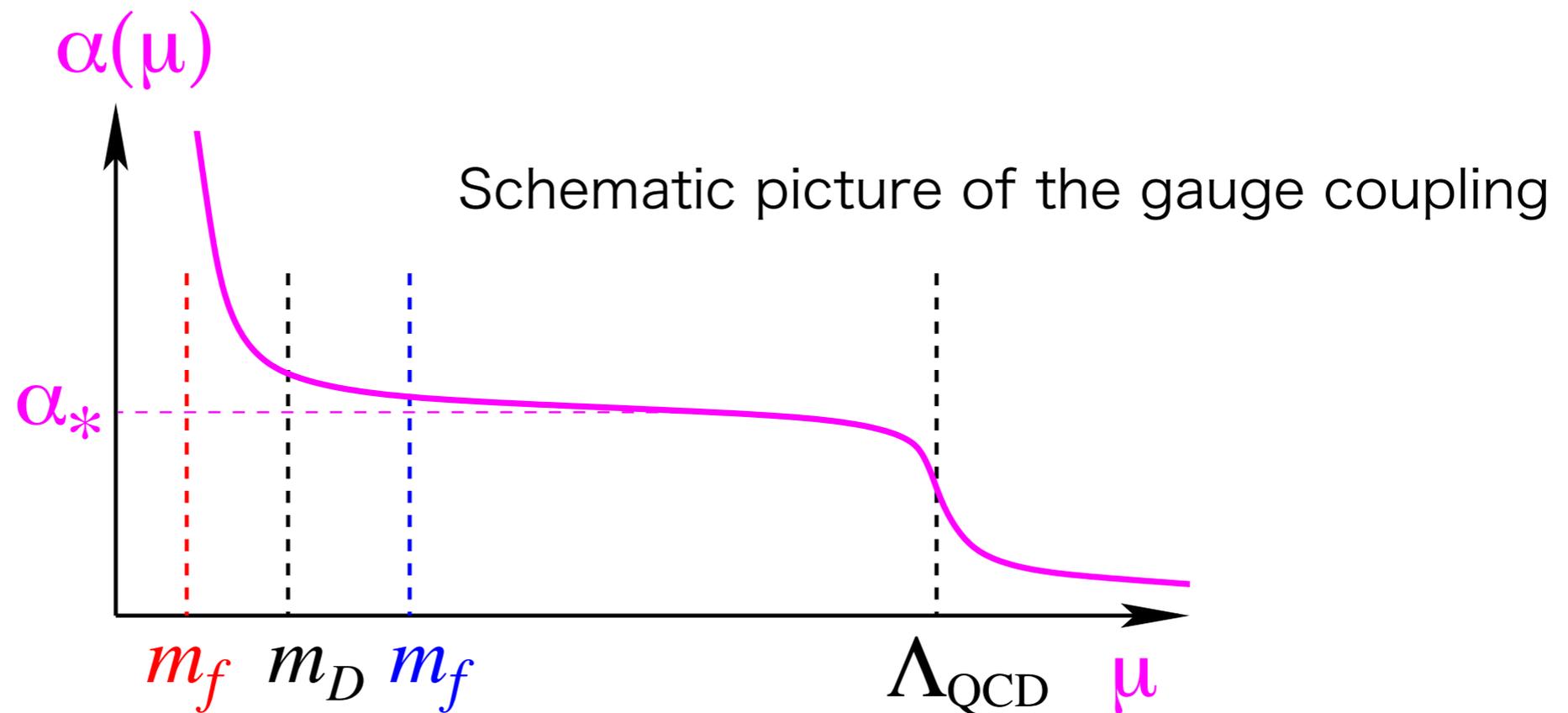
not favored as WTC (model building)

Thus, we investigate  $N_f=8$  QCD.  
strong coupling dynamics and non-perturbative



Lattice simulation of  $N_f=8$  QCD

# What is the signal of walking?



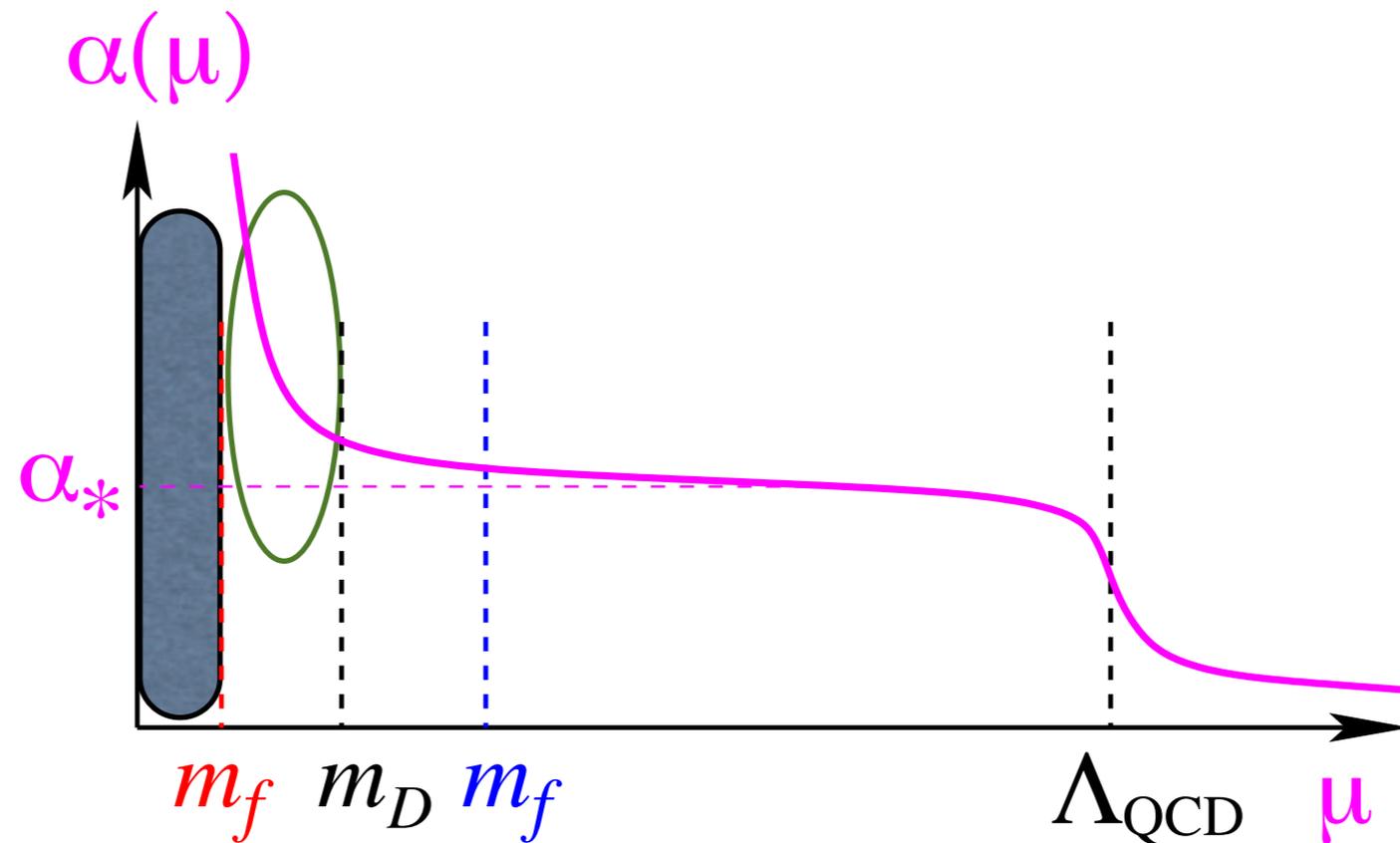
in Gauge Theories

⇒ **Spectrum?** Heuristically

S  $\chi$  SB and/or conformal ?

What is the signal of walking?

Case-1: probe  $m_f \ll m_D \rightarrow S \chi$  SB-like

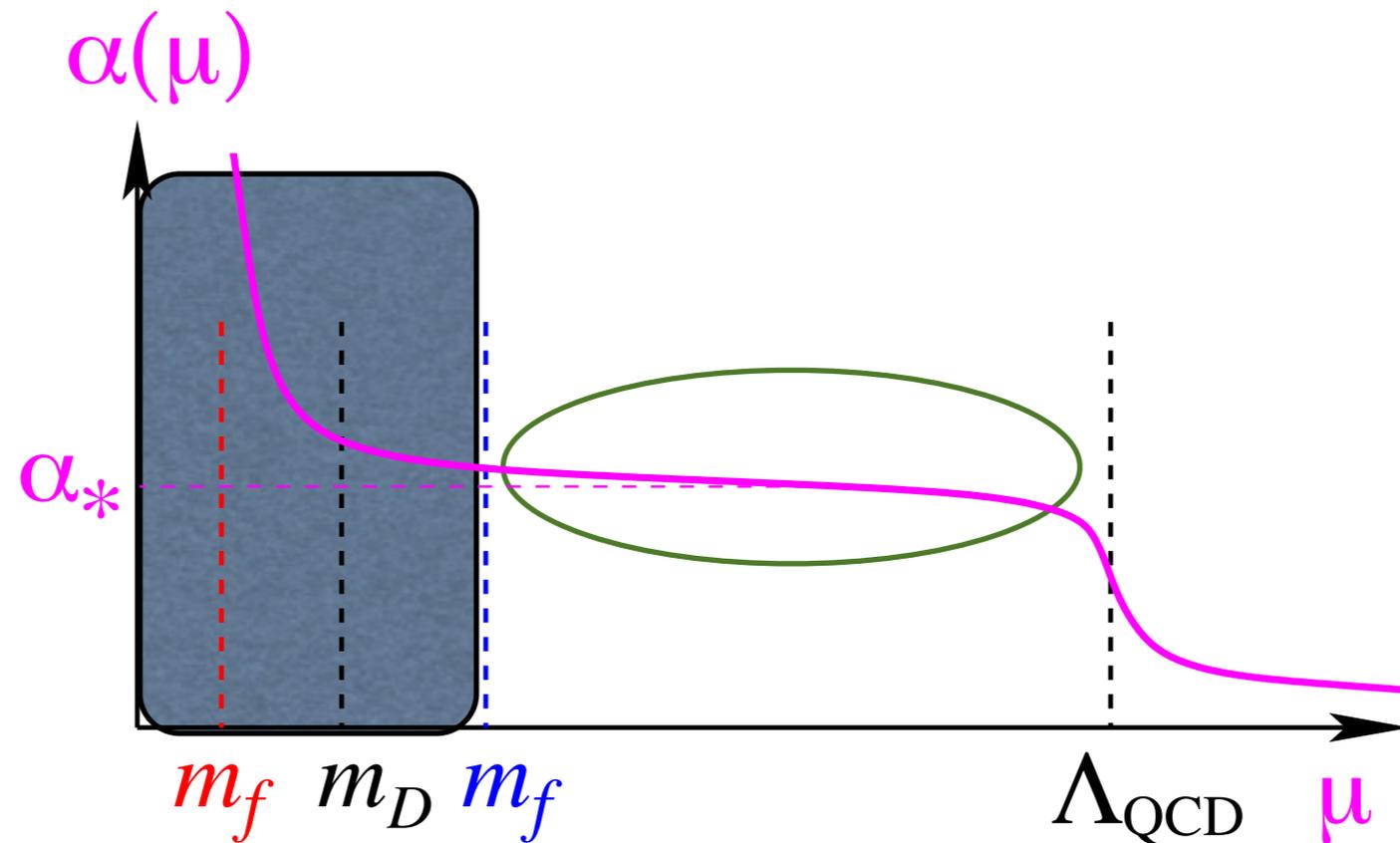


$\Rightarrow$  **Spectrum?**

$S \chi$  SB and/or conformal ?

# What is the signal of walking?

Case-2: probe  $m_f \gg m_D \rightarrow$  conformal-like



$\Rightarrow$  **Spectrum?**

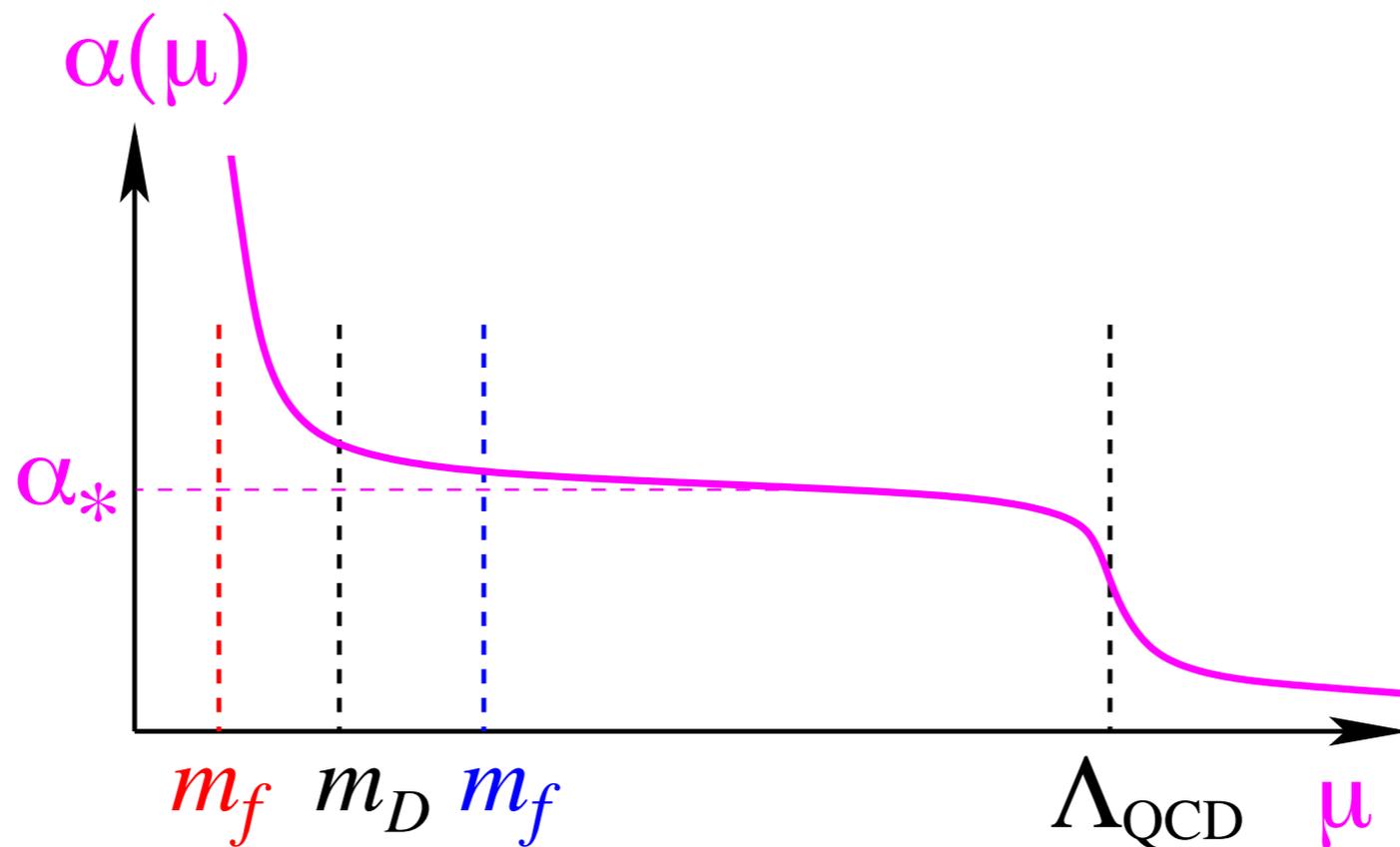
$S \chi$  SB and/or conformal ?

Case-1: probe  $m_f \ll m_D \rightarrow S \chi SB$ -like

(ChPT near  $\chi$ -limit)

Case-2: probe  $m_f \gg m_D \rightarrow$  conformal-like

(Hyperscaling w/ or w/o mass corrections)



(Also, LSD collab. paper, arXiv:1405.4752)

$\Rightarrow$  **Spectrum?**

$S \chi SB$  and/or conformal ?

## **2. Walking signals in $N_f=8$ QCD**

# Simulation for $N_f=8$ (same setup with $N_f=12$ )

lattice action (Hybrid Monte-Carlo simulation)

- Tree-level Symanzik gauge action
- Highly Improved Staggered Quarks = **HISQ**  
(without tadpole improvement and mass correction in Naik term)

## ★ parameter set

- $\beta (\equiv 6/g^2) = \mathbf{3.8}$ ,  $V=L^3 \times T$ ,  $T/L=4/3$  fixed.

V	$12^3 \times 16$	$18^3 \times 24$	$24^3 \times 32$	$30^3 \times 40$	$36^3 \times 48$	$42^3 \times 56$
mf	0.01~0.16	0.04~0.1	0.02~0.1	0.03,0.04~0.07	0.015,0.02,0.03	0.012

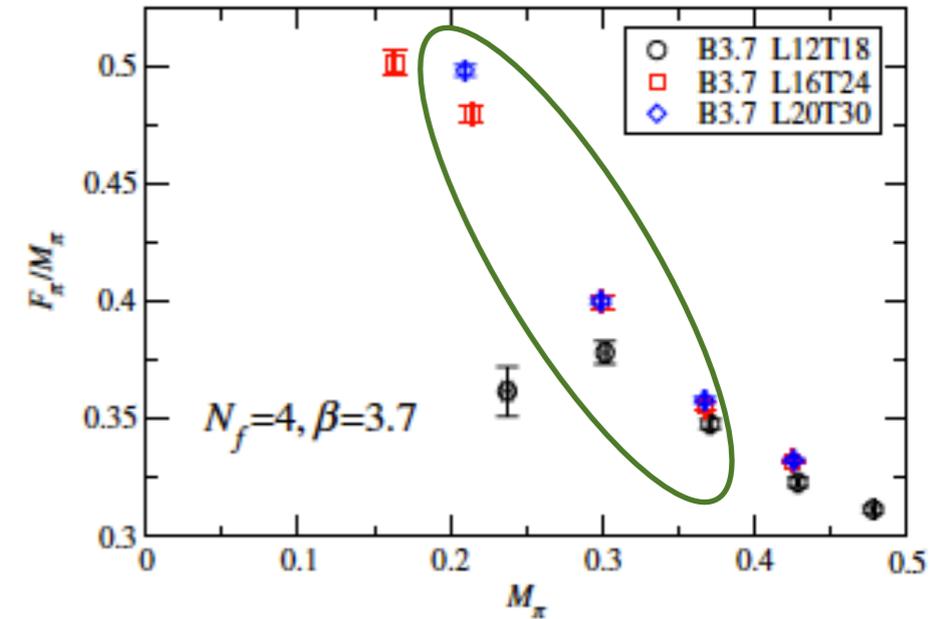
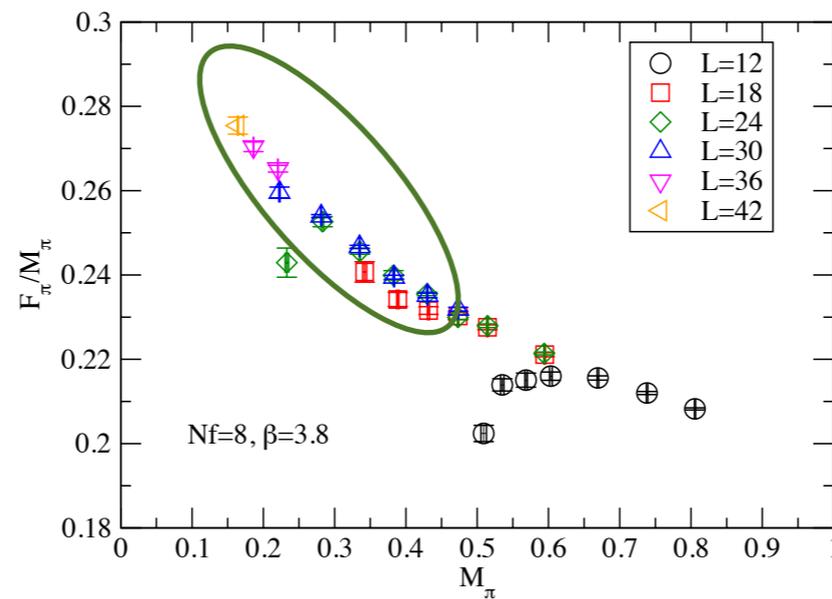
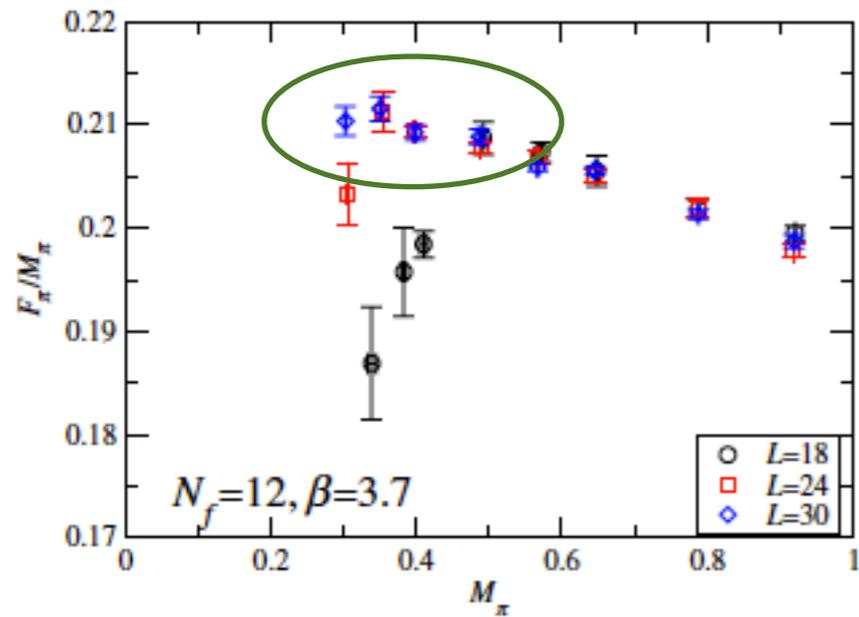
(updated) (new)

★ Gauge configurations for scalar measurement

★ Measurements (P+AP method  $\Rightarrow$  double size in T-dir.)

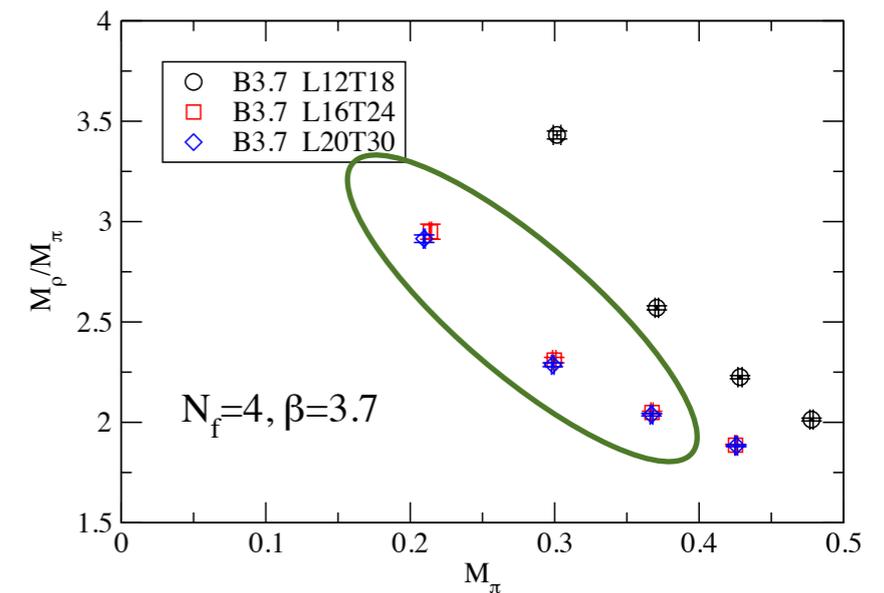
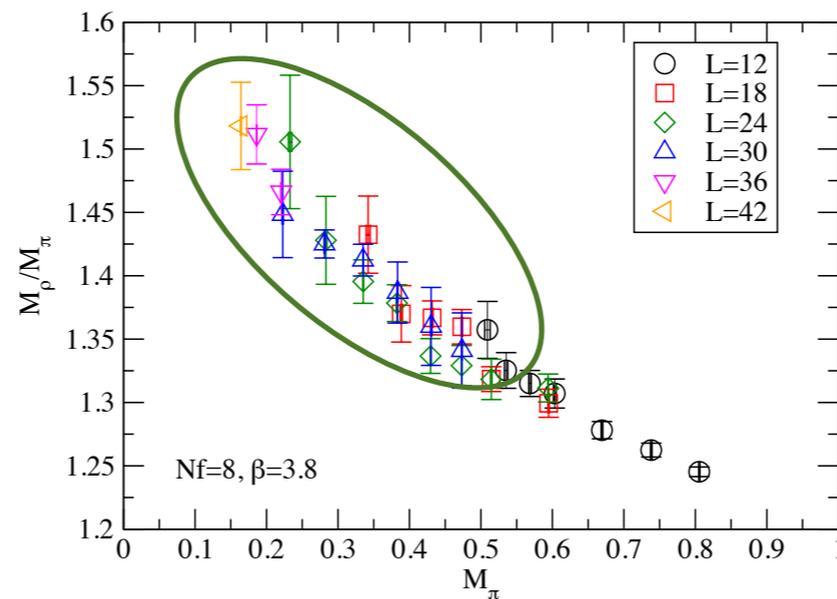
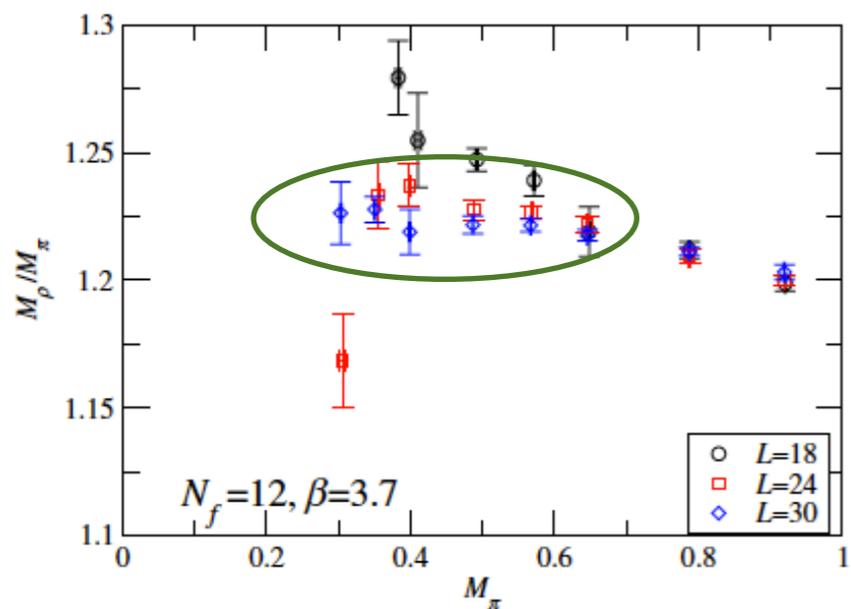
- $M_\pi$ ,  $F_\pi$ ,  $M_\rho$ , chiral condensate
- analysis for  $M_\pi L > 6$

# $F_\pi/M_\pi$ for $N_f=12, 8$ and $4$



(updated: 2013→2014)

# $M_\rho/M_\pi$ for $N_f=12, 8$ and $4$

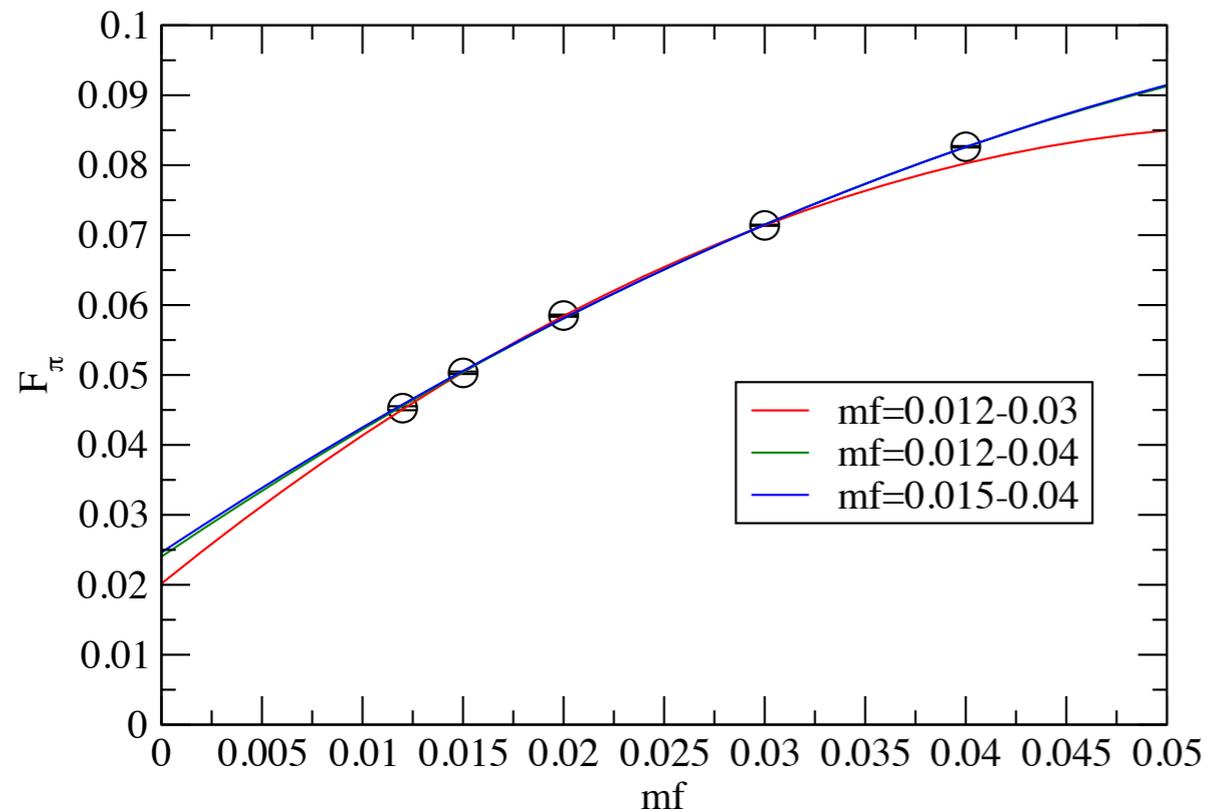


(updated: 2013→2014)

ChPT analysis in small mf region

# $F_\pi$ vs $mf$ with quadratic-func. fit : $y=C_0+C_1*mf+C_2*mf^2$

$F_\pi$  vs  $mf$ , and quadratic fit



red :  $C_0=0.0202(13)$ ,  $\chi^2/\text{dof}=0.60$  (dof=1)

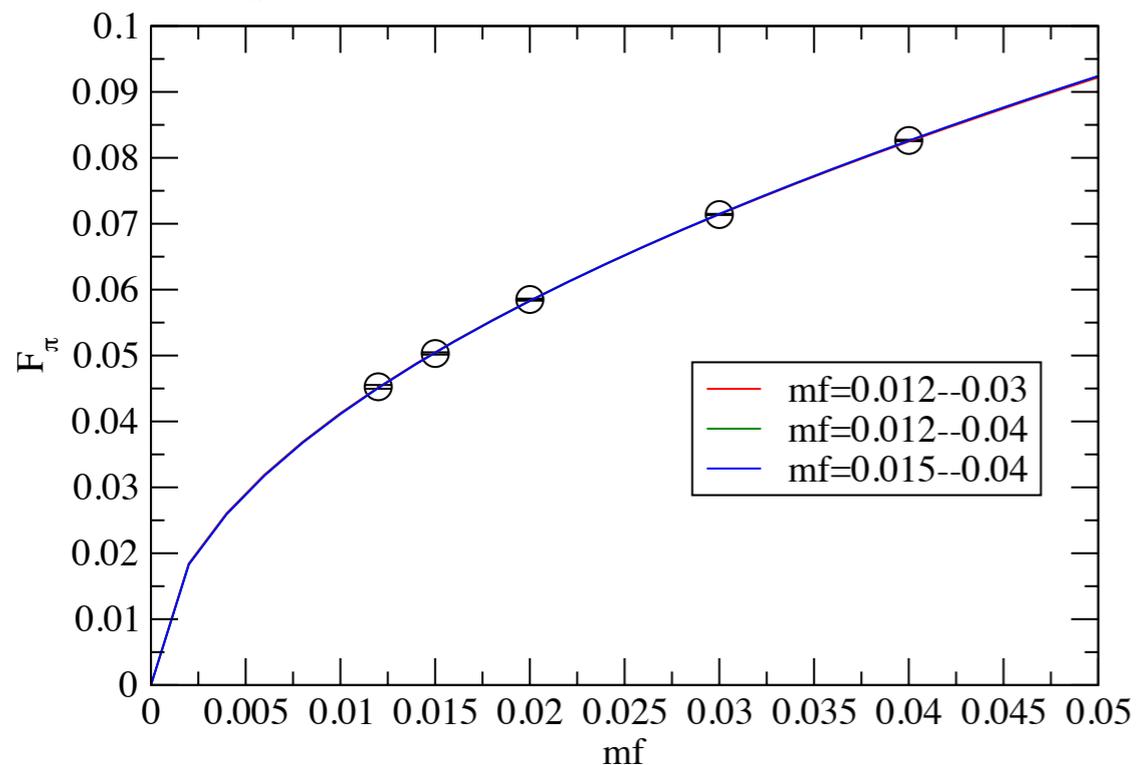
green:  $C_0=0.0240(5)$ ,  $\chi^2/\text{dof}=5.9$  (dof=2)

blue :  $C_0=0.0246(7)$ ,  $\chi^2/\text{dof}=9.4$  (dof=1)

ChPT seems to be good in  $0.012 \leq mf \leq 0.03$ .

# $F_\pi$ vs $mf$ with power-func. fit (hyperscaling) : $y=D_1*mf^\alpha$

$F_\pi$  vs  $mf$  and the fitting of power func. w/o intercept



red :  $\alpha=0.500(4)$ ,  $\chi^2/\text{dof}=1.22$  (dof=2)

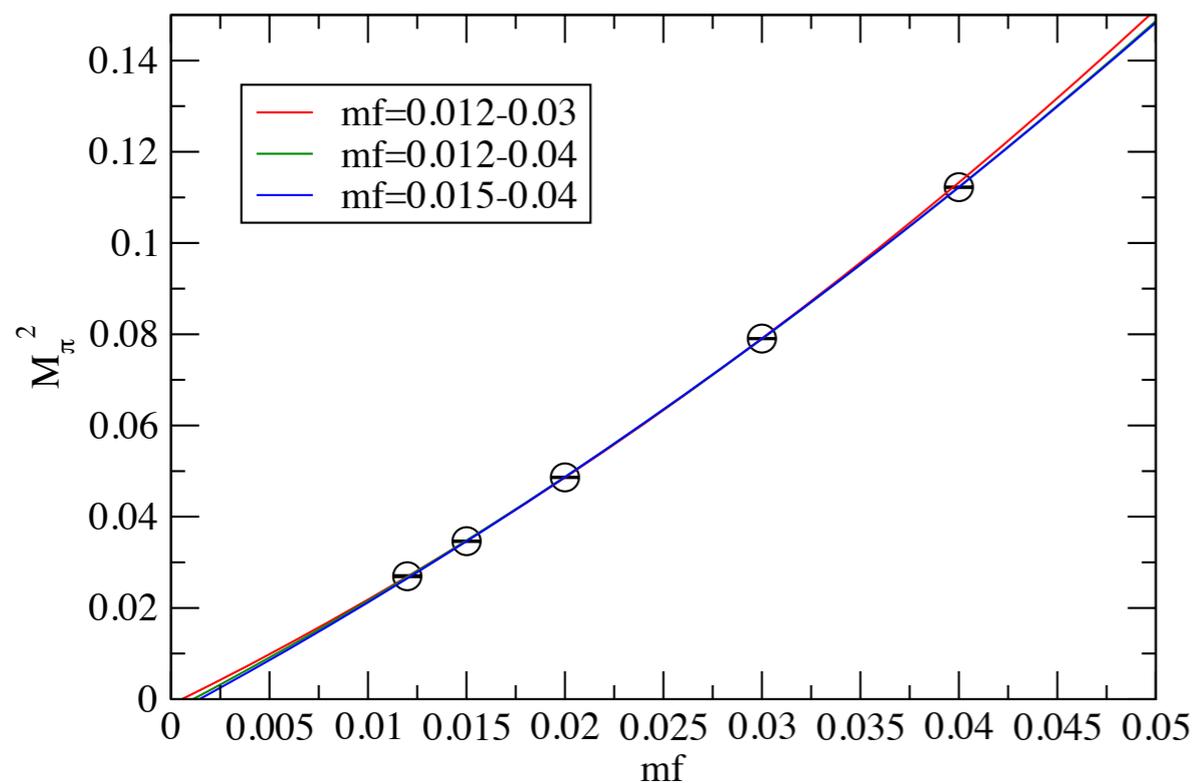
green:  $\alpha=0.0502(3)$ ,  $\chi^2/\text{dof}=1.07$  (dof=3)

blue :  $\alpha=0.0503(3)$ ,  $\chi^2/\text{dof}=1.46$  (dof=2)

HS fit seems to be good in  $0.04 \leq mf$ .

# $M_\pi^2$ vs mf with quadratic-func. fit

$M_\pi^2$  vs mf, and quadratic fit



red :  $C_0 = -0.0012(11)$ ,  $\chi^2/\text{dof} = 0.79$  (dof=1)

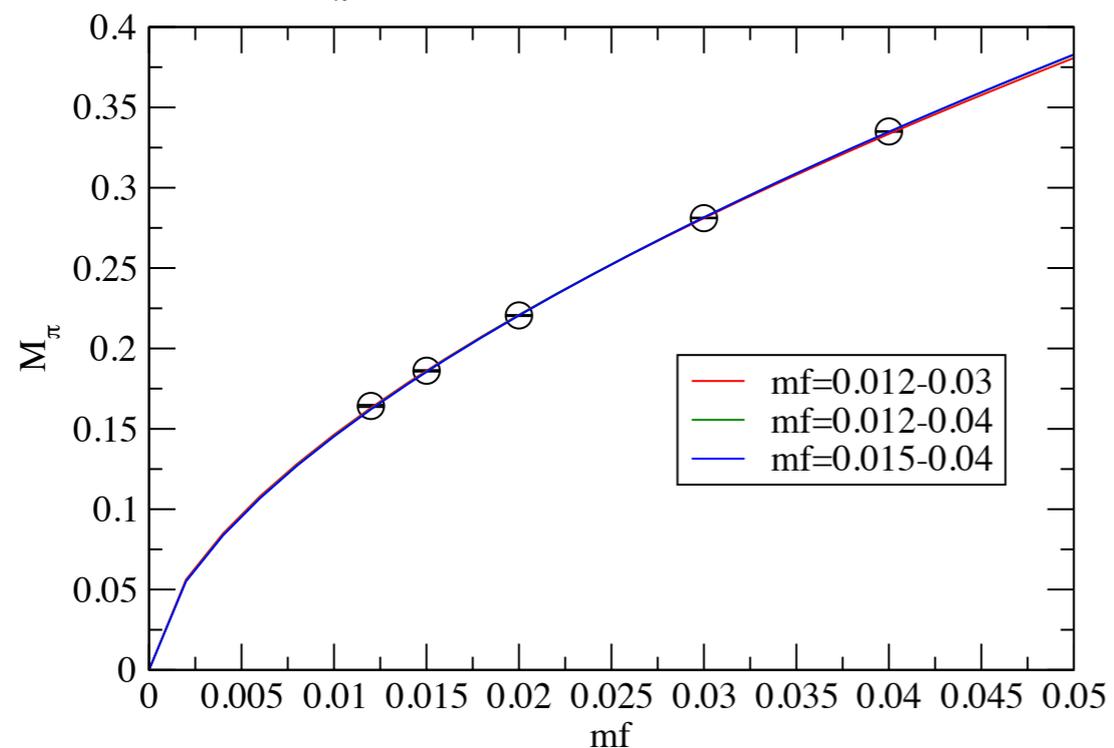
green:  $C_0 = -0.0025(6)$ ,  $\chi^2/\text{dof} = 1.4$  (dof=2)

blue :  $C_0 = -0.0034(8)$ ,  $\chi^2/\text{dof} = 0.24$  (dof=1)

ChPT in  $0.012 \leq mf \leq 0.03$ .

# $M_\pi$ vs mf with power-func. fit (hyperscaling) : $y = D_1 * mf^\alpha$

$M_\pi$  vs mf, and power-func. fit w/o const.



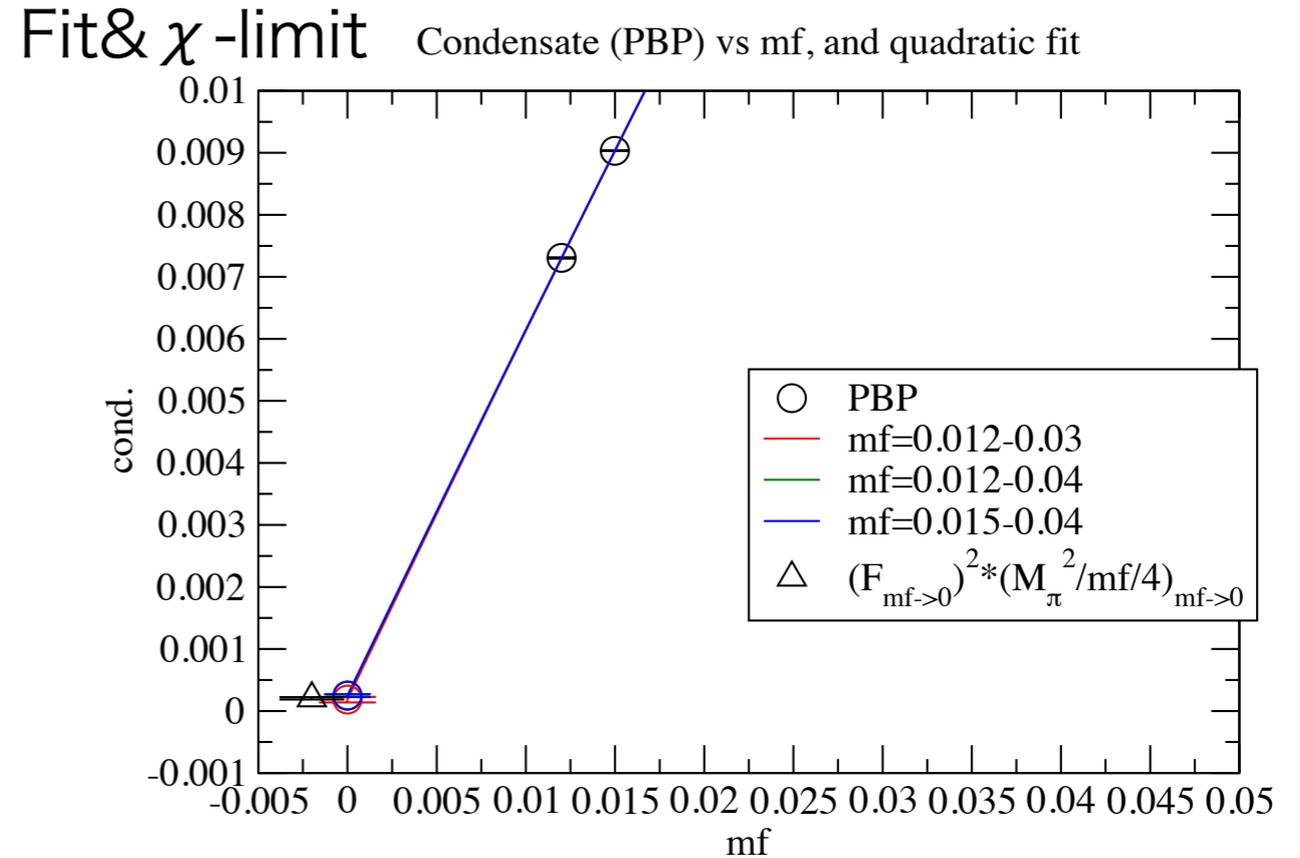
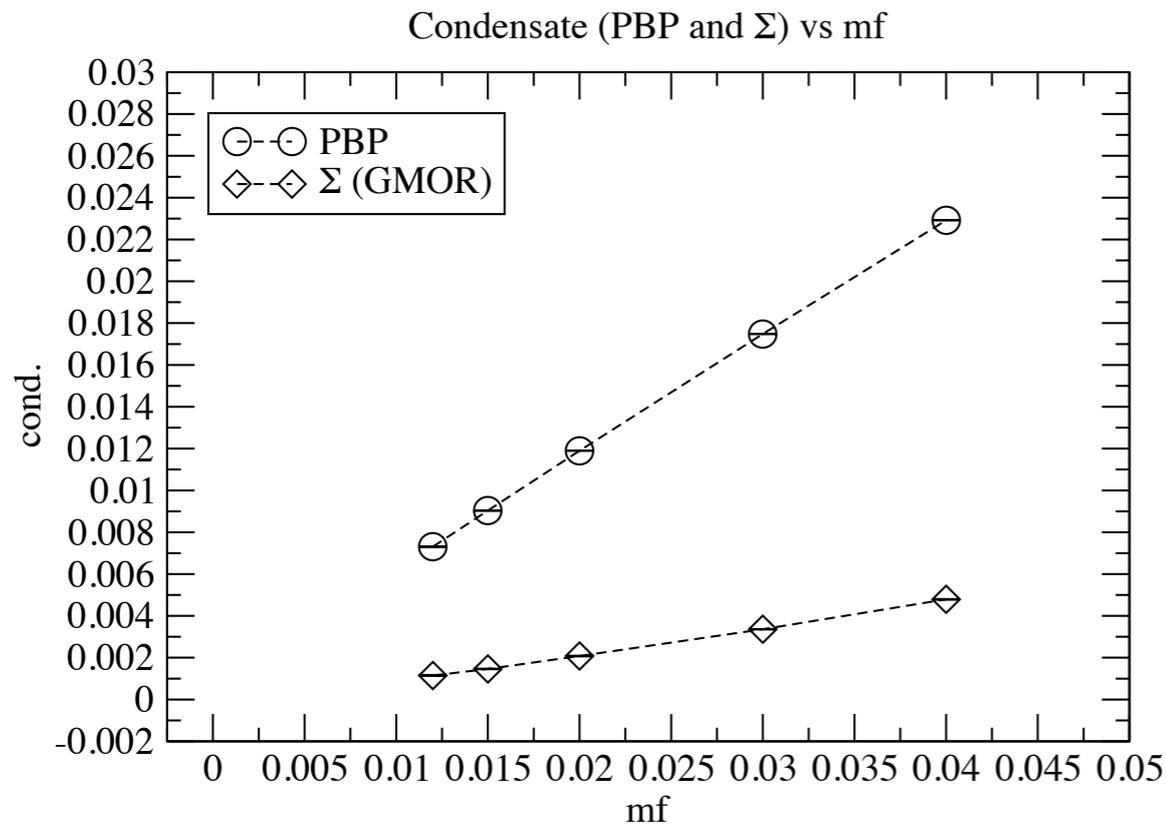
red :  $\alpha = 0.594(3)$ ,  $\chi^2/\text{dof} = 2.5$  (dof=2)

green:  $\alpha = 0.600(2)$ ,  $\chi^2/\text{dof} = 5.0$  (dof=3)

blue :  $\alpha = 0.602(2)$ ,  $\chi^2/\text{dof} = 2.9$  (dof=2)

# Condensate (PBP and $\Sigma$ ) and quadratic-func. fit

$$\text{PBP} = \text{Tr}[\text{Prop}(x,x)/4], \quad \Sigma = f_\pi^2 M_\pi^2 / mf / 4$$



PBP

red :  $C_0 = 0.00018(5)$ ,  $\chi^2/\text{dof} = 0.89$

green:  $C_0 = 0.00024(2)$ ,  $\chi^2/\text{dof} = 1.61$

blue :  $C_0 = 0.00025(2)$ ,  $\chi^2/\text{dof} = 3.18$

$$\Sigma = F_\pi M_\pi^2 / mf / 4$$

$$(F_\pi)^2_{mf \rightarrow 0} * (M_\pi^2 / mf / 4)_{mf \rightarrow 0} = 0.00021(2)$$

In  $0.012 \leq mf \leq 0.03$ , consistent value

Hyperscaling analysis in the heavy mf region

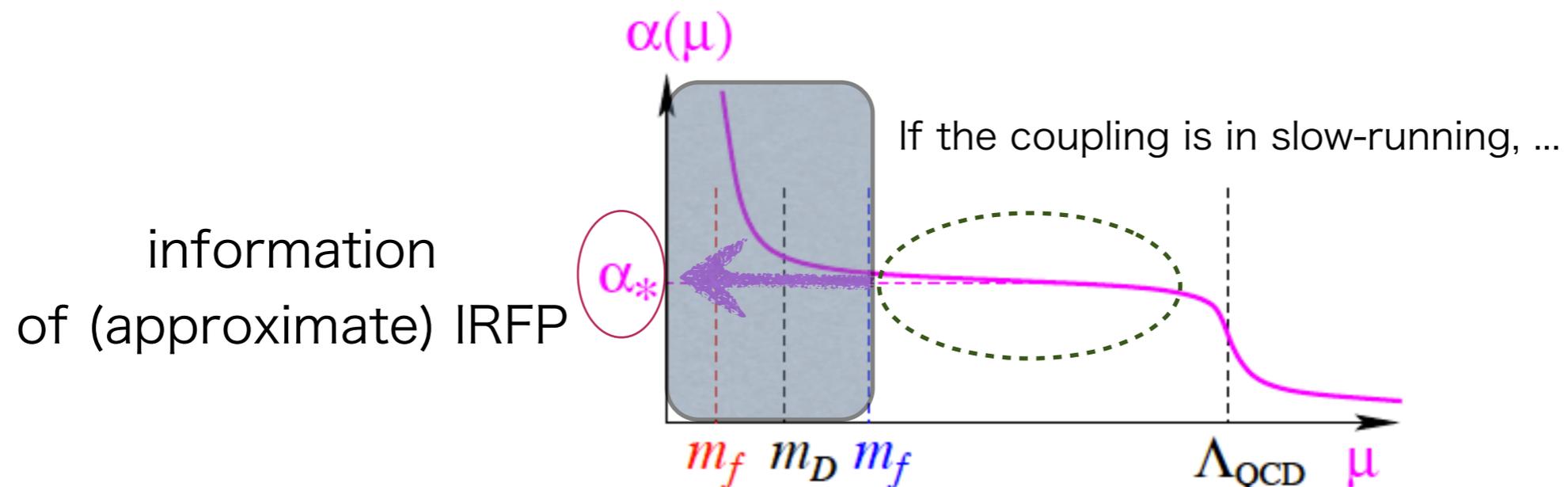
# Finite size Hyperscaling analysis in a intermediate mf

Conformal  $\rightarrow$  Finite size Hyperscaling behavior with universal  $\gamma$

$$LM_H = \mathcal{F}_H(x), LF_H = \mathcal{G}_F(x) \quad (\text{DeGrand, Del Debbio et al.})$$

$$x \equiv L m^{1/1+\gamma} \quad (\text{near the } \chi\text{-limit})$$

However, in our walking scenario,



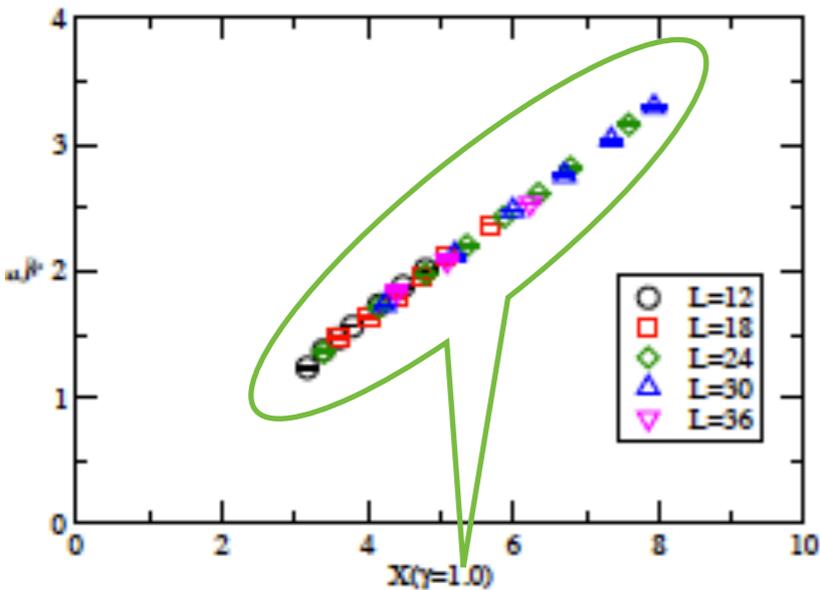
# Finite size Hyperscaling analysis

Comformal  $\rightarrow$  Finite size Hyperscaling behavior with universal  $\gamma$

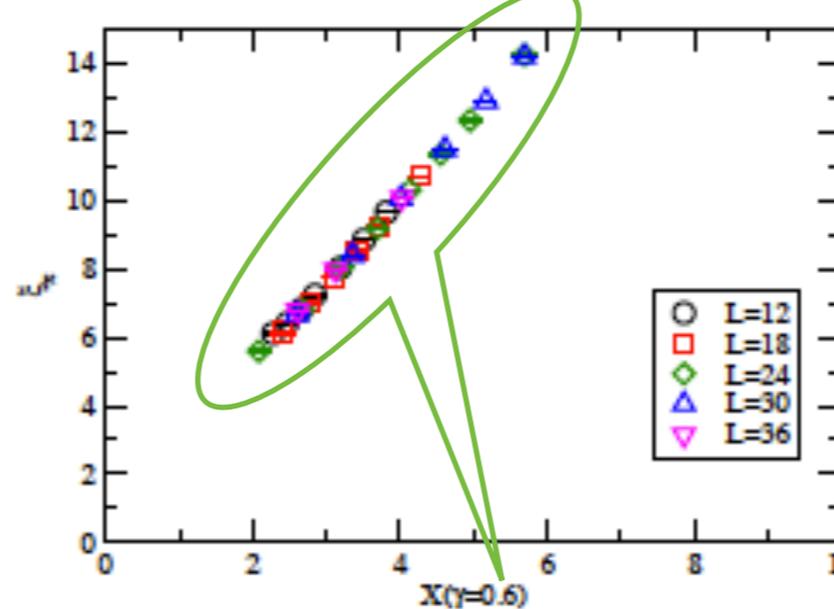
$$LM_H = \mathcal{F}_H(x), LF_H = \mathcal{G}_F(x)$$

$$x \equiv L m^{1/1+\gamma}$$

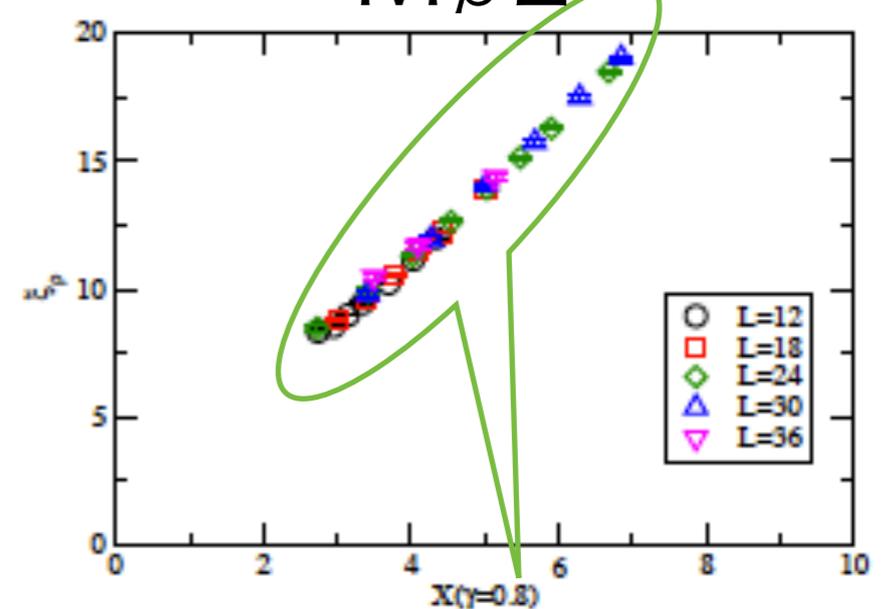
$F_\pi L$



$M_\pi L$



$M_\rho L$



$$\gamma(F_\pi) \sim 1.0, \gamma(M_\pi) \sim 0.6, \gamma(M_\rho) \sim 0.8$$

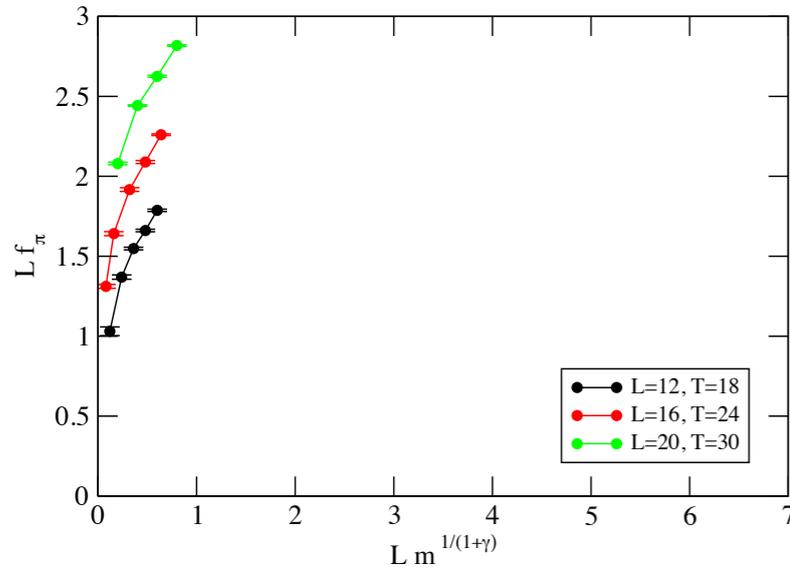
different from  $N_f=4$  and 12

non-universal  $\gamma$  in the naive FSHS analysis

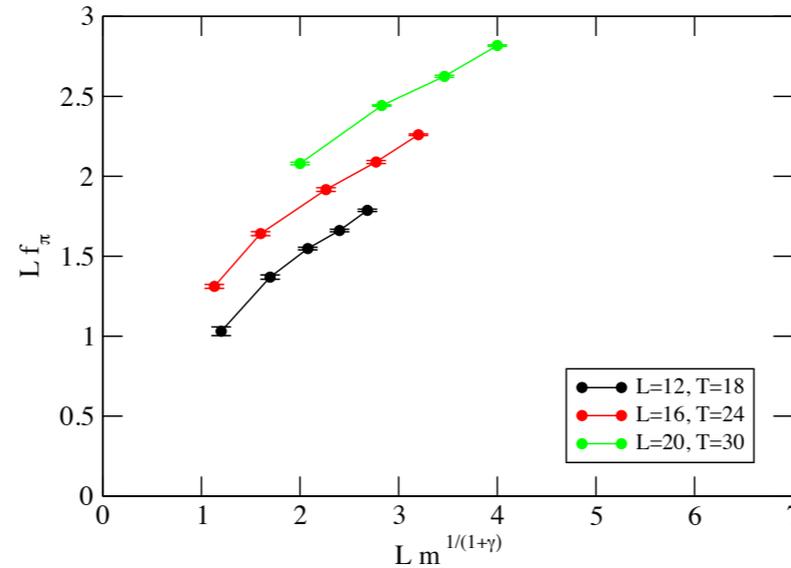
# Comparison with Nf=4

: hyperscaling of  $F_\pi$  (in  $S_\chi$  SB)

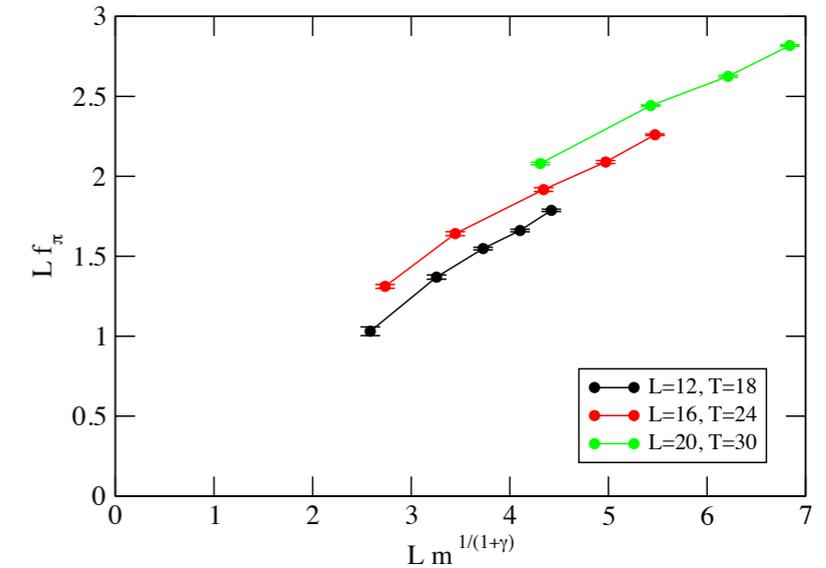
$\beta=3.7, \gamma = 0.0$



$\beta=3.7, \gamma = 1.0$

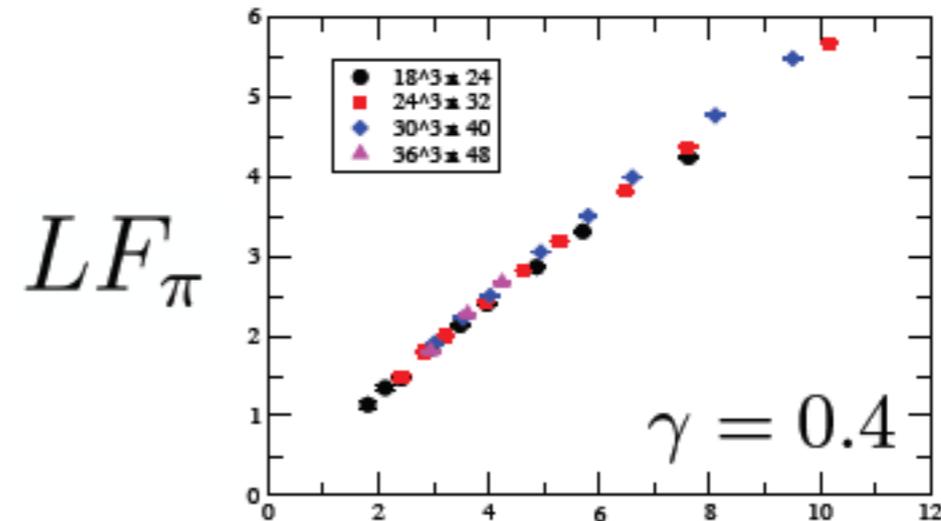
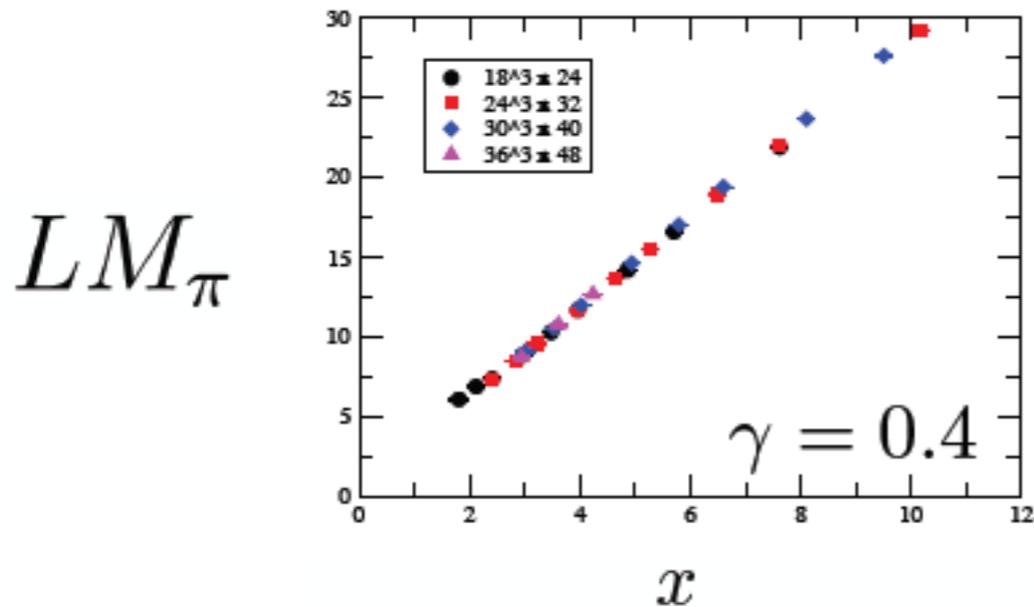


$\beta=3.7, \gamma = 2.0$



$F_\pi$  : no data collapsing for  $0 < \gamma < 2$

# Comparison with Nf=12: universal $\gamma$



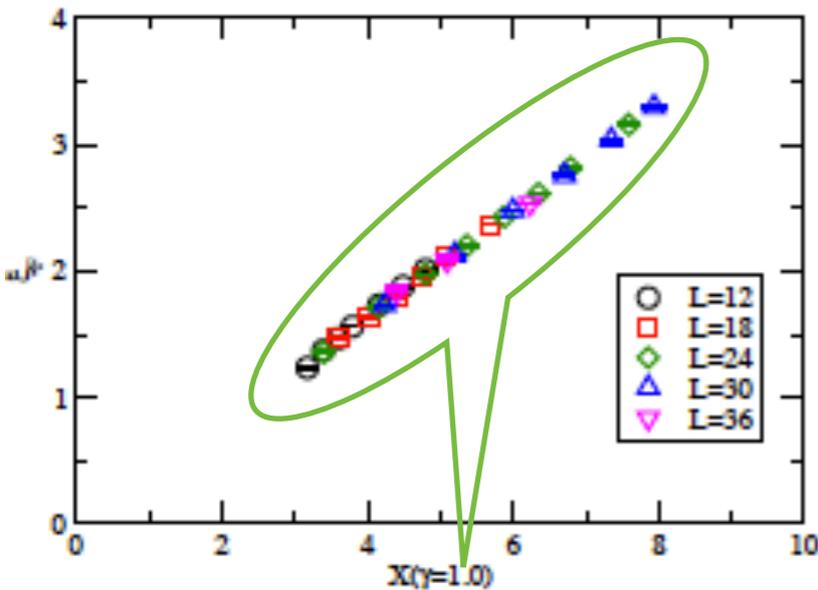
# Finite size Hyperscaling analysis

Comformal  $\rightarrow$  Finite size Hyperscaling behavior with universal  $\gamma$

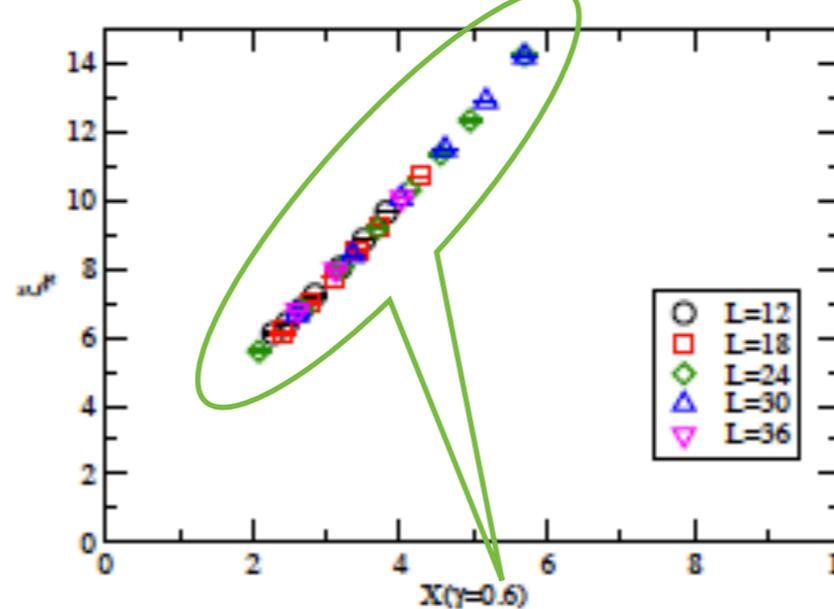
$$LM_H = \mathcal{F}_H(x), LF_H = \mathcal{G}_F(x)$$

$$x \equiv L m^{1/1+\gamma}$$

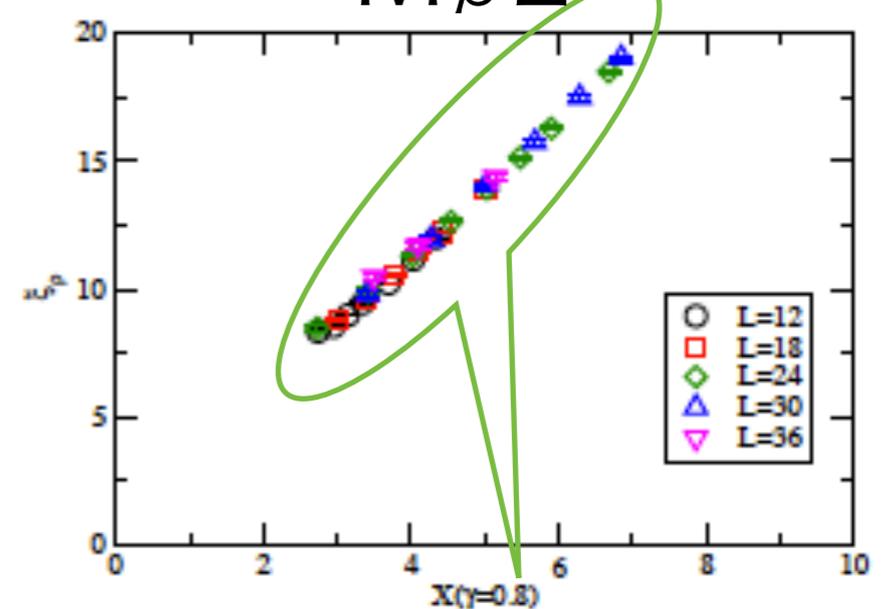
$F_\pi L$



$M_\pi L$



$M_\rho L$



$$\gamma(F_\pi) \sim 1.0, \gamma(M_\pi) \sim 0.6, \gamma(M_\rho) \sim 0.8$$

different from  $N_f=4$  and 12

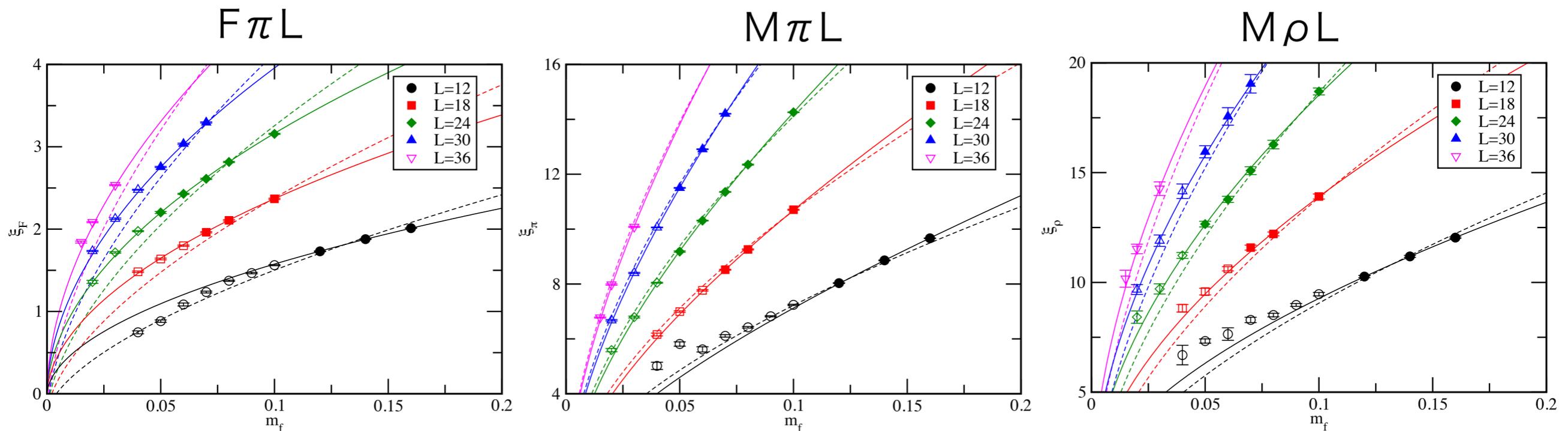
non-universal  $\gamma$  in the naive FSHS analysis

# Simultaneous fit of Hyperscaling with mass corrections

$$\xi_H = C_0^H + C_1^H X + C_2^H L m_f^\alpha \quad \text{in the middle region of } m_f \geq 0.05 \text{ and } \xi_\pi (=M_\pi L) \geq 8$$

(Schwinger-Dyson eq. analysis with large mass: e.g.  $\alpha = (3 - 2\gamma)/(1 + \gamma)$ )

$\xi_H (=M_H L)$  vs  $m_f$  (not  $X$ ):  $\alpha = 1$  fixed ( $\Downarrow$  example)



In this case, the mass correction works well with  $\gamma = 0.874(25)$ ,  $\chi^2/\text{dof} = 0.75$ ,  $\text{dof} = 32$ . (solid line)

- In various trials of this analysis:  $\gamma = 0.78 - 0.93 \sim 1$

# Walking signals.

+updated (this talk)

- $N_f=8$  is (still) consistent with ChPT(S  $\chi$  SB) in the small  $mf$  region.  
( $0.015 \leq mf \leq 0.04 \Rightarrow 0.012 \leq mf \leq 0.03$ )

- In the  $\chi$ -limit.,  $F=0.031(1)(+2-10) \Rightarrow 0.0202(13)(+54-67)$ ,

$$M_\rho/(F/\sqrt{2})=7.7(1,5) \Rightarrow 8.5(2.1) , \text{Cond}=0.00052(5) \Rightarrow 0.00021(2).$$

- The expansion parameter  $\chi=O(1)$  of ChPT in the smallest  $mf$  (self-consistent), in contrast to  $N_f=12$ .

$$\chi = N_f \left( \frac{M_\pi}{4\pi F} \right)^2$$

- In the region of  $mf \geq 0.05$ , hyperscaling is seen. (different from  $N_f=4$ )
- non-universal  $\gamma$  for each observable in the naive finite-size hyperscaling (different from  $N_f=12$ )
- Simultaneous fit of hyperscaling with mass correction (from SD analysis) universal  $\gamma \sim 1$ . [remnant of conformality]

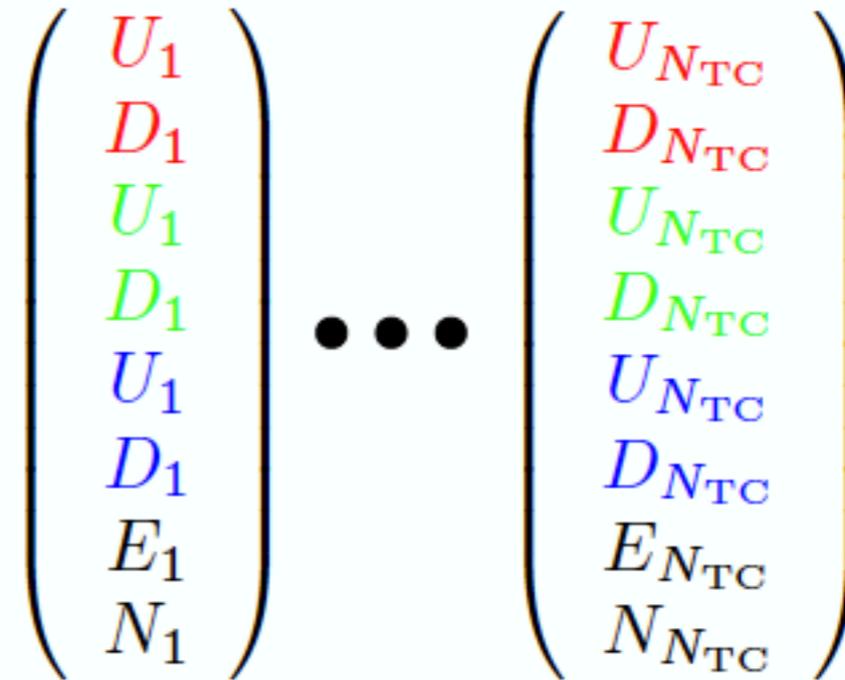
### **3. Light composite scalar in $N_f=8$**

# SU( $N_{TC}$ ) gauge theory with 8 fundamental fermions

Techni-fermion condensate is the origin of the EWSB

$$f_\pi^2 = v_{EW}^2 / N_D = (123 \text{ GeV})^2$$

$N_D$  : # of weak doublet  
 $\parallel$   
 4



Techni-fermions

126 GeV Higgs  
 = Composite Higgs  
 = Techni-Dilaton

$$m_H \simeq f_\pi$$

Dilaton ChPT (DChPT)  
 S. Matsuzaki and K. Yamawaki, arXiv:1311.3784  
 $m_\sigma^2 = d_0 + d_1 m_\pi^2$ , where  $d_1 = \frac{(3 - \gamma)(1 + \gamma)}{4} \frac{N_F F_\pi^2}{F_\sigma^2} \sim 1$

For Scalar measurement;

$m_f$	$L^3 \times T$	$N_{\text{cf}}[N_{\text{st}}]$	$m_\sigma$	$m_\pi$	$F_\pi$
0.015	$36^3 \times 48$	3200[2]	0.155(21) $\binom{0}{41}$	0.1861(4)*	0.0503(2)*
0.02	$36^3 \times 48$	5000[1]	0.190(17) $\binom{39}{0}$	0.2205(3)*	0.0585(1)*
0.02	$30^3 \times 40$	8000[1]	0.201(21) $\binom{0}{60}$	0.2227(9)	0.0578(2)
0.03	$30^3 \times 40$	16500[1]	0.282(27) $\binom{24}{0}$	0.2812(2)*	0.07140(9)*
0.03	$24^3 \times 32$	36000[2]	0.276(15) $\binom{6}{0}$	0.2832(14)	0.0715(4)
0.04	$30^3 \times 40$	12900[3]	0.365(43) $\binom{17}{0}$	0.3349(3)*	0.0826(1)*
0.04	$24^3 \times 32$	50000[2]	0.322(19) $\binom{8}{0}$	0.3353(7)	0.0823(2)
0.04	$18^3 \times 24$	9000[1]	0.228(30) $\binom{0}{16}$	0.3421(29)	0.0823(5)
0.06	$24^3 \times 32$	18000[1]	0.46(7) $\binom{12}{0}$	0.4295(6)	0.1012(3)
0.06	$18^3 \times 24$	9000[1]	0.386(77) $\binom{12}{0}$	0.4317(15)	0.0999(5)

TABLE I: Simulation parameters for  $N_f = 8$  QCD at  $\beta = 3.8$ .  $N_{\text{cf}}(N_{\text{st}})$  is the total number of gauge configurations (Markov chain streams). The second error of  $m_\sigma$  is a systematic error coming from the fit range. The values for  $m_\pi$  and  $F_\pi$  are from Ref. [10], but the ones with (\*) have been updated.

## Difficulty of flavor-singlet scalar meson

- Flavor non-singlet scalar meson  $S_{NS}(t) = \sum_{\vec{x}} \bar{\psi}_a(\vec{x}, t) \psi_b(\vec{x}, t)$  ( $a \neq b$ )

$$\langle 0 | S_{NS}(t) S_{NS}^\dagger(0) | 0 \rangle = \left\langle \text{Diagram} \right\rangle = -C(t)$$

c.f.  $m_\pi, F_\pi$  from non-singlet pseudoscalar

$O(100)$  configurations  $\times O(1) D^{-1}[U](x, y)$

- Flavor-singlet scalar meson  $S(t) = \sum_{\vec{x}} \bar{\psi}_a(\vec{x}, t) \psi_a(\vec{x}, t)$

$$\langle 0 | S(t) S^\dagger(0) | 0 \rangle = -C(t) + (N_f/4) D(t) \text{ (disconnected)}$$

$$D(t) = \left\langle \text{Diagram 1} \quad \text{Diagram 2} \right\rangle - \left\langle \text{Diagram 3} \right\rangle^2$$

Much harder but essential for flavor-singlet

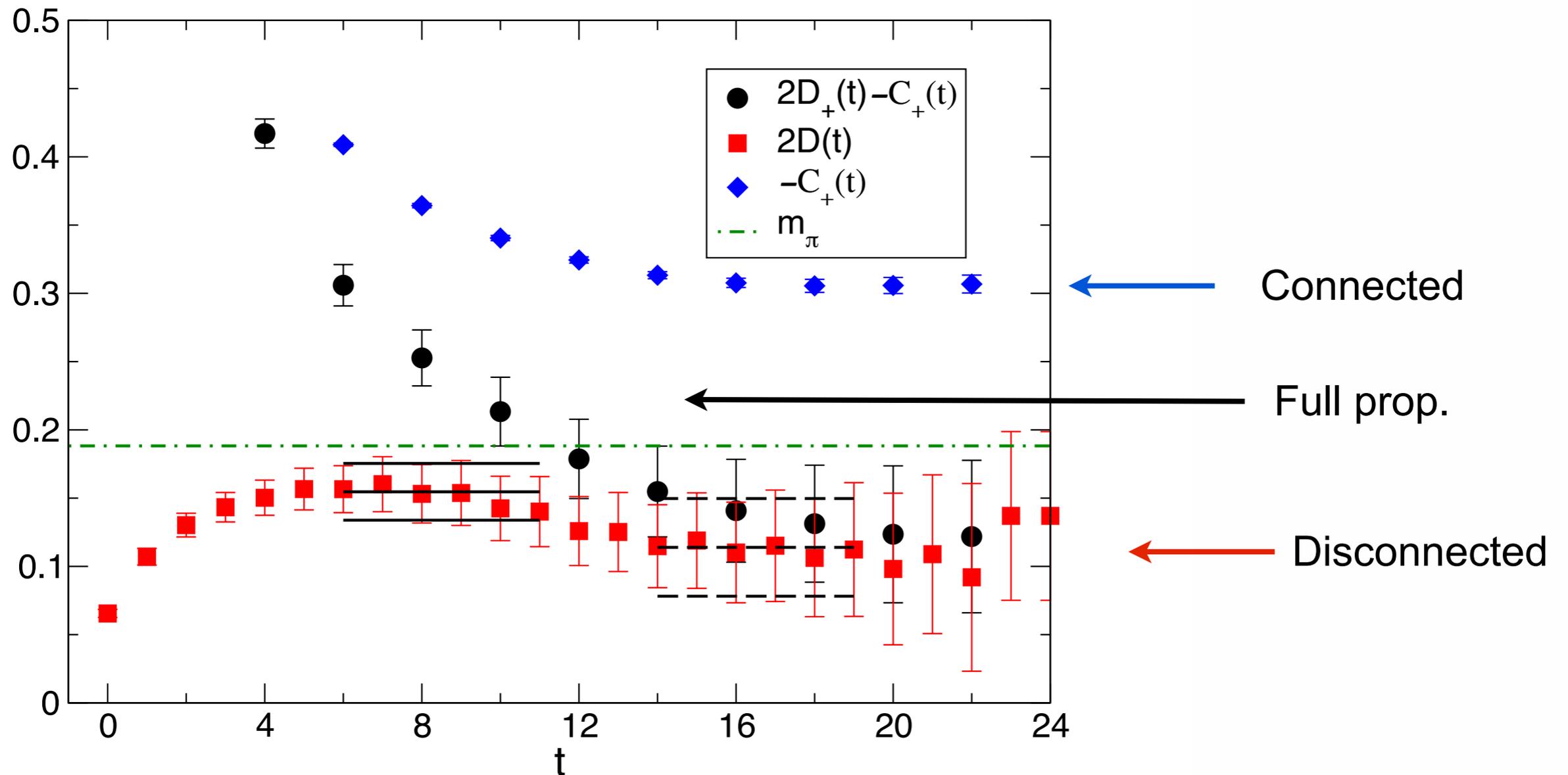
$O(10000)$  configurations  $\times O(10) D^{-1}[U](x, x)$

using noise reduction method

'97 Venkataraman and Kilcup

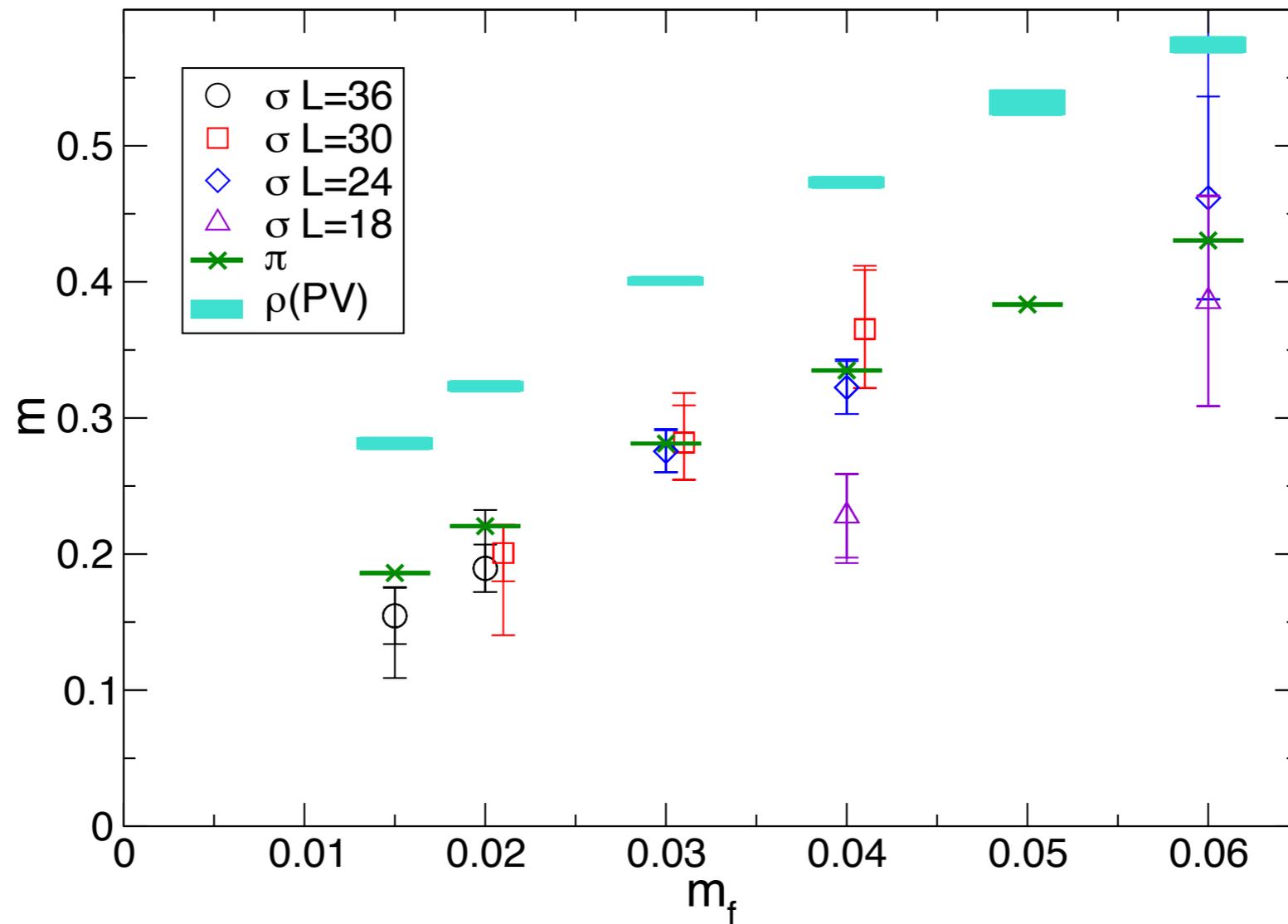
# Effective mass (mf=0.06, L=24, 14000config., Nf=8)

$$m_{\text{eff}}(t) = \log(C_H(t)/C_H(t+1)) \xrightarrow{t \gg 1} m_H$$



**good signal !!**

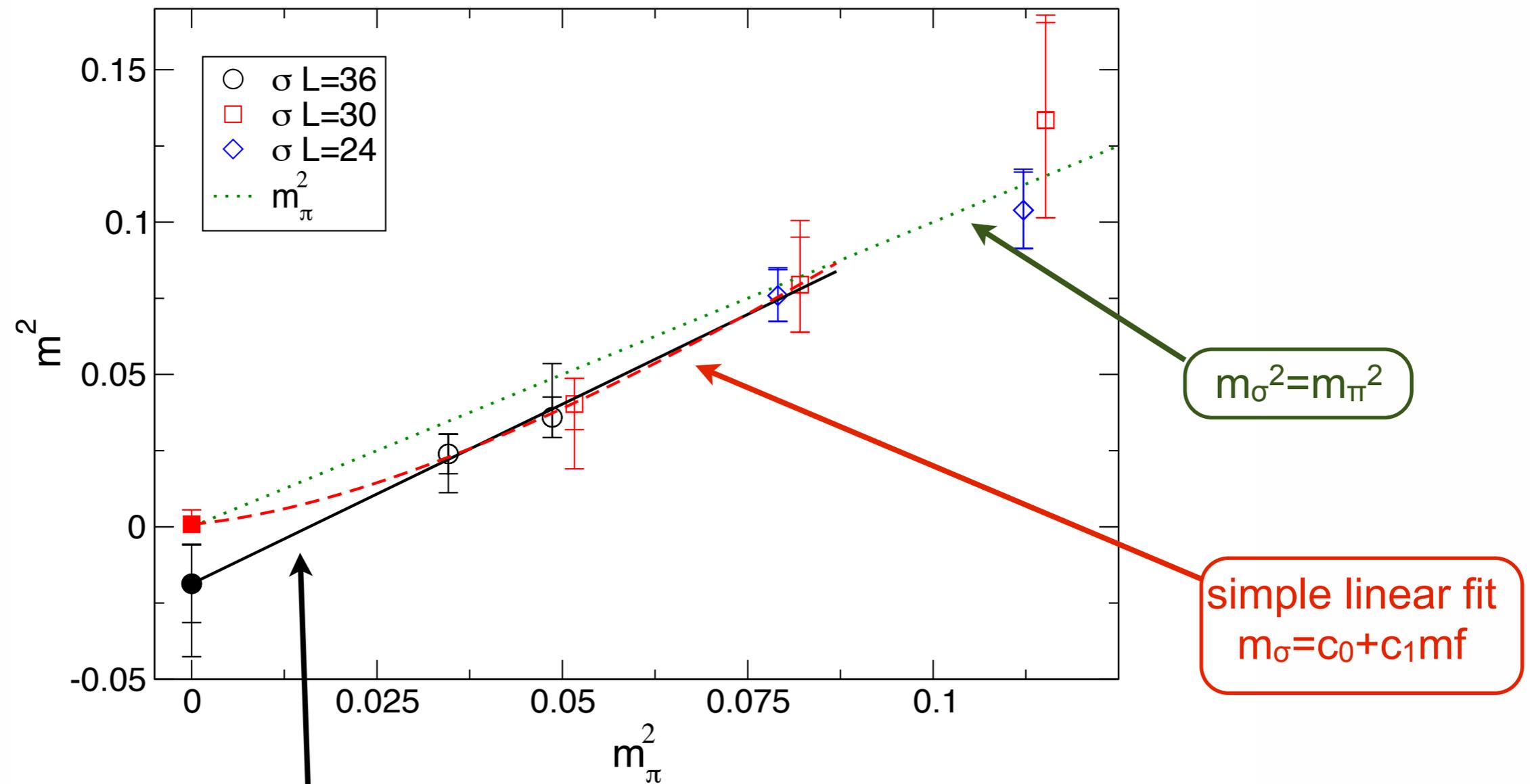
# Results: Nf=8



$M_\rho > M_\sigma \sim M_\pi$

Nf=8 QCD is in sharp contrast to the real-life QCD

# Fit to the data in the small mf region where ChPT works



Dilaton ChPT (DChPT) fit

S. Matsuzaki and K. Yamawaki, arXiv:1311.3784

$$m_\sigma^2 = d_0 + d_1 m_\pi^2, \text{ where } d_1 = \frac{(3 - \gamma)(1 + \gamma)}{4} \frac{N_F F_\pi^2}{F_\sigma^2} \sim 1$$

# Comparison of $m_\sigma$ in $N_f = 8$ with $m_{\text{Higgs}}$

In chiral limit:

$F/\sqrt{2} = 123$  GeV; One-family model (four-doublet fermions)

- Simple linear fit

$$m_\sigma / (F/\sqrt{2}) = 2.0(2.7) \left(\frac{8}{5}\right)$$

- ChPT with spontaneous scale symmetry breaking

Dilaton ChPT (DChPT) fit

S. Matsuzaki, K. Yamawaki arXiv:1311.3784

$$m_\sigma^2 = d_0 + d_1 m_\pi^2$$

lattice result:  $d_0 = -0.019(13) \left(\frac{3}{9}\right)$

( $f_\pi = F/\sqrt{2}$ ,  $F \doteq 0.02$  input)

Nearly consistent with

$$m_\sigma^2 \simeq f_\pi^2 (\simeq 0.0002)$$

though uncertainty is still large

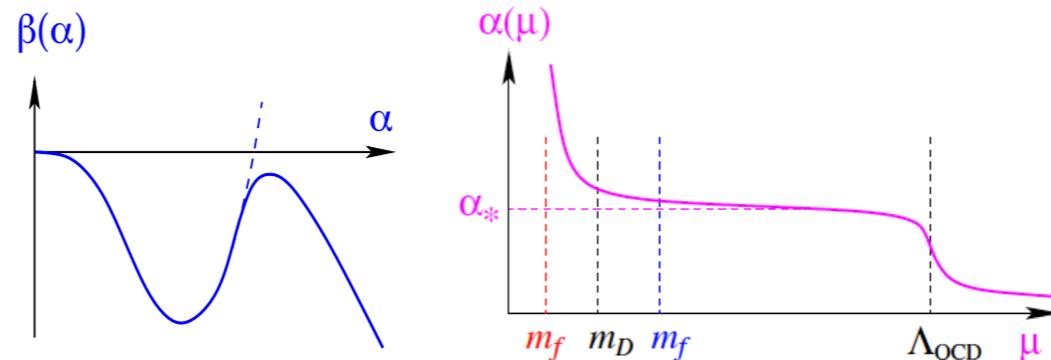
→ very large error for very small value → not inconsistent

⇒ Possibility of reproduce  $m_{\text{Higgs}}$

# Summary

- ◆ SU(3) gauge theories with 4, 12, 16 and **8 HISQ quarks**.  
Preliminary (data updated: 2013 → 2014)
- ◆ Nf=8; possibility of  $S\chi SB$  in the small mass region of our simulation and the remnant of the conformality in the middle region of mf with  $\gamma \sim 1$ . (In contrast to Nf=4 and 12 cases.)
- ◆ **Light Scalar** (flavor singlet scalar meson, techni-dilaton)

Nf=8 → Candidate for Walking dynamics



Phenomenologically preferable?

## In Progress:

- ◆ Simulation on larger volumes at lighter masses
- ◆ Finite Size Effect (one-volume:  $m_f=0.015$  on  $L=36$ ,  $m_f=0.012$  on  $L=42$ )
- ◆ Finite Size Scaling
- ◆ Lattice spacing dependence (cont' limit)  $\leftarrow$  many  $\beta$
- ◆ Spectroscopy ( $M_{\text{glueball}}$ ,  $M_{\text{scalar}}$ ,  $M_{\text{baryon}}$ ,  $M_{\text{meson}}$ ,  $F_\sigma$ , S-param., string tension, etc.)

$M_{\text{scalar}}$ ,  $M_{\text{glueball}}$ , string tension

- ◆  $M_{\text{flavor-singlet light scalar}}$

$N_f=12$

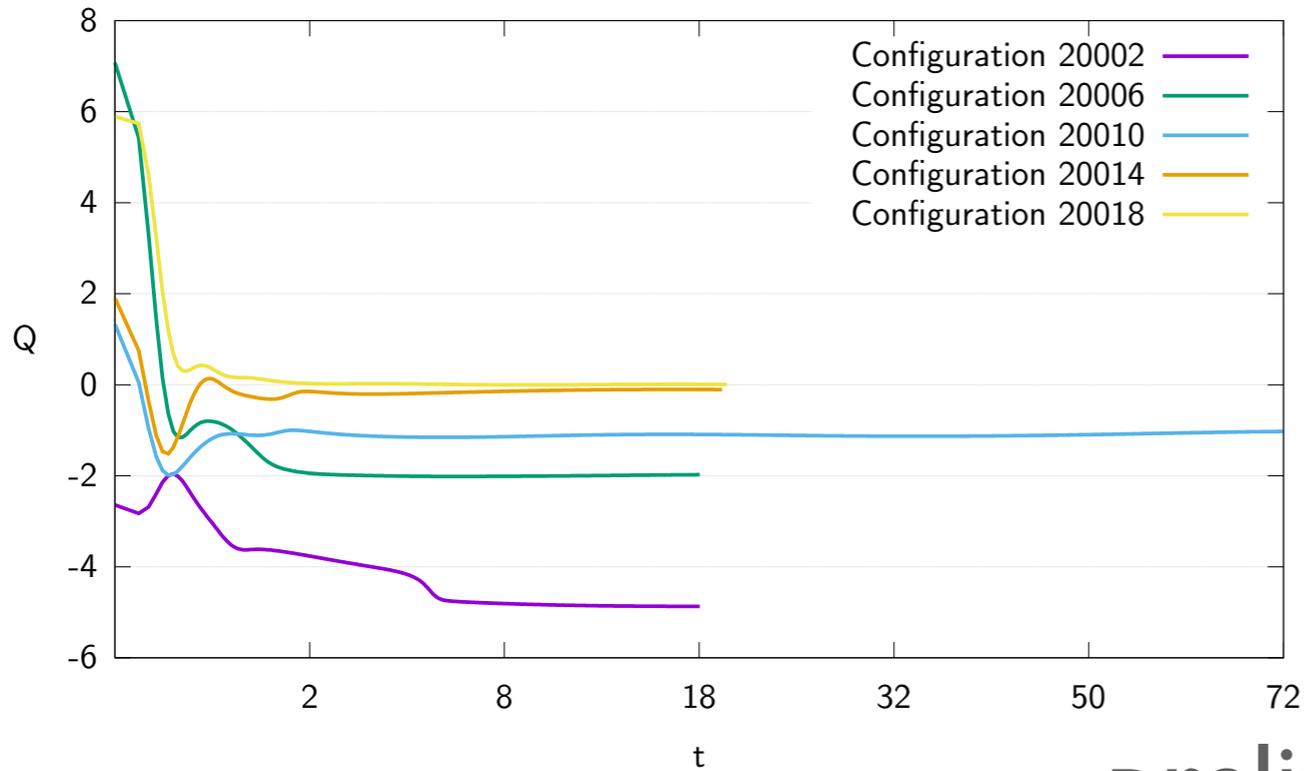
- ◆ Topological charge and susceptibility
- ◆ Gradient flow
- ◆ Eigenvalues

Ohki's talk (Next!)

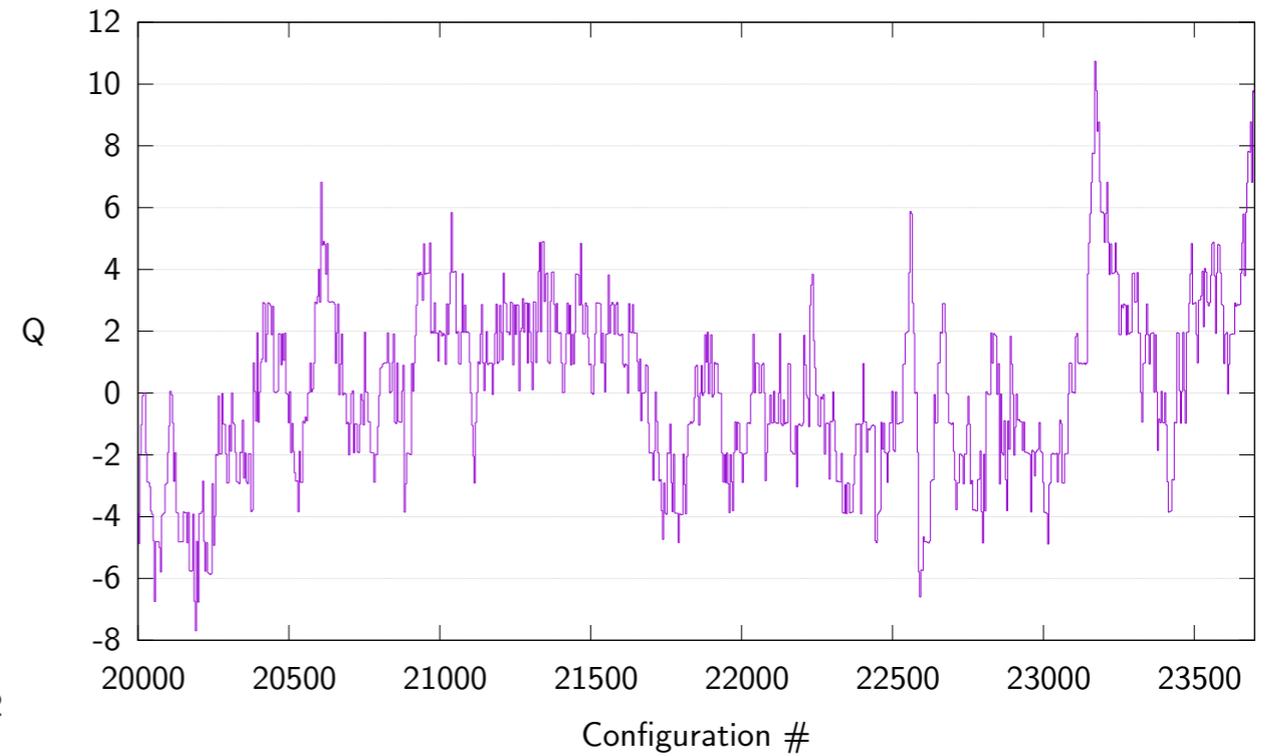
Aoki-Rinaldi's poster

# In $N_f=8$ at $mf=0.06$ on $L=24$

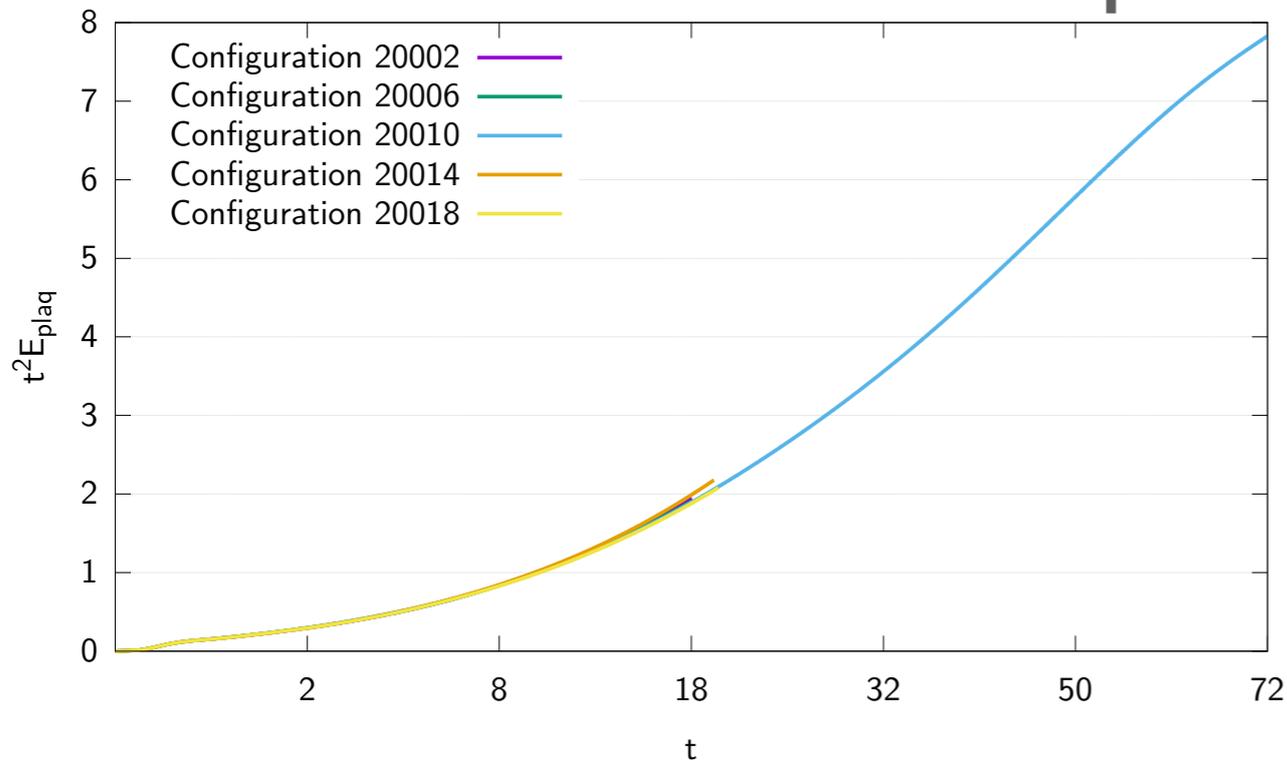
## $Q_{\text{top}}$ vs Symanzik flow time



## $Q_{\text{top}}$ vs #trajectory



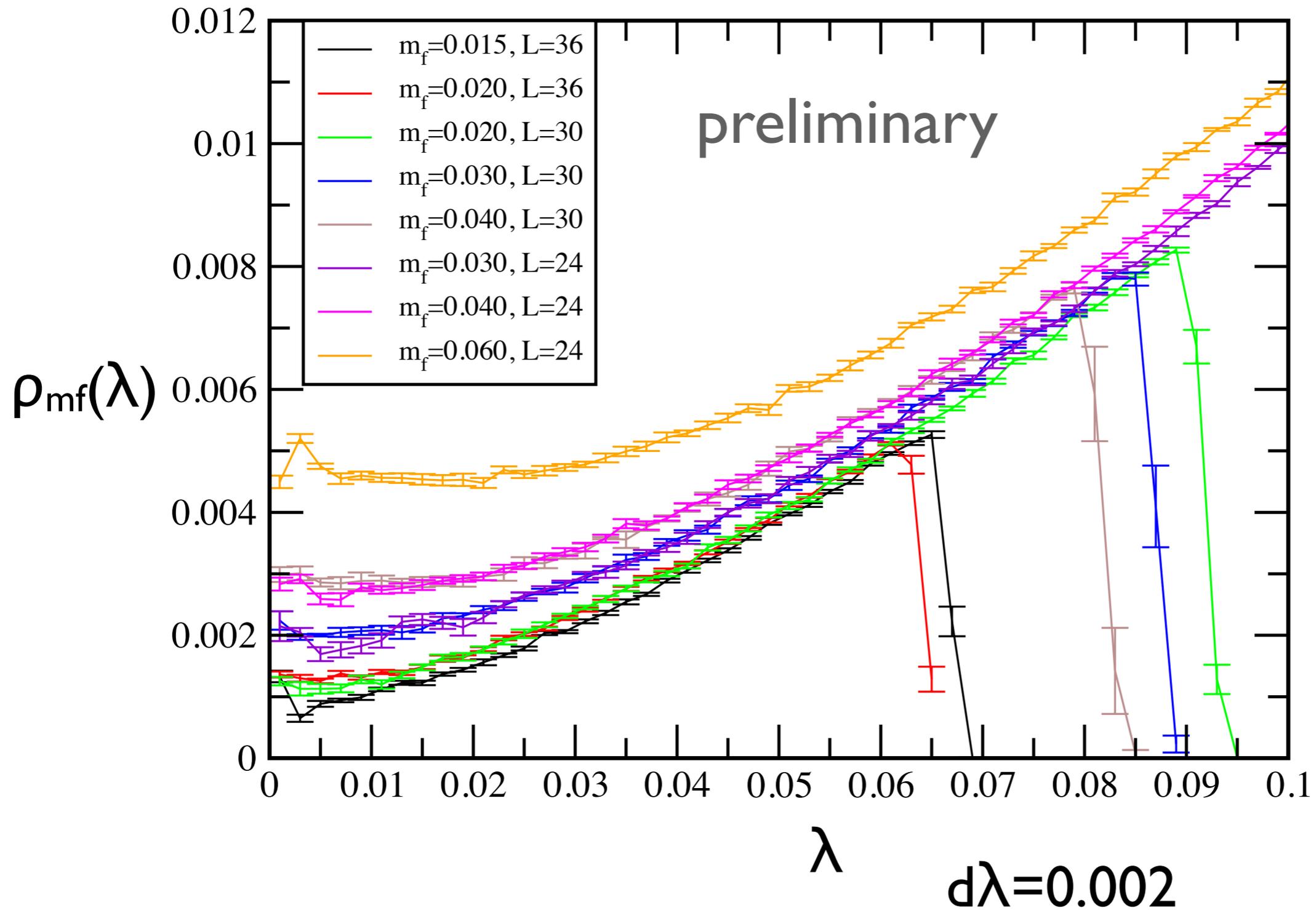
preliminary



## $t^2 E_{\text{plaq}}$ vs flow time

$\rho_{mf}(\lambda)$  in  $N_f=8$

$\lambda = \sqrt{EV(D_{\text{HISQ}}^\dagger D_{\text{HISQ}})}$ , for  $D_{\text{HISQ}}^\dagger D_{\text{HISQ}} + m^2$



$\Rightarrow \Sigma, \gamma, \dots, \text{RMT}, \dots$ , in the near future

## In Progress:

- ◆ Simulation on larger volumes at lighter masses
- ◆ Finite Size Effect (one-volume:  $m_f=0.015$  on  $L=36$ ,  $m_f=0.012$  on  $L=42$ )
- ◆ Finite Size Scaling
- ◆ Lattice spacing dependence (cont' limit)  $\leftarrow$  many  $\beta$
- ◆ Spectroscopy ( $M_{\text{glueball}}$ ,  $M^{\text{"scalar"}}$ ,  $M_{\text{baryon}}$ ,  $M_{\text{meson}}$ ,  $F_\sigma$ , S-param., string tension, etc.)

$M^{\text{"scalar"}}$ ,  $M_{\text{glueball}}$ , string tension

- ◆  $M^{\text{"flavor-singlet light scalar"}}$

$N_f=12$

- ◆ Topological charge and susceptibility
- ◆ Gradient flow
- ◆ Eigenvalues

Ohki's talk (Next!)

Aoki-Rinaldi's poster

Thank you

Backup

# HISQ with Nf=8:

effective mass for the lowest  $m_f (=0.015)$   
on the largest size ( $L=36$ )

$$M_{\pi}^{\text{PS}} = M_{\pi}^{\text{SC}}, M_{\rho}^{\text{PV}} = M_{\rho}^{\text{VT}}$$

→ good flavor symmetry

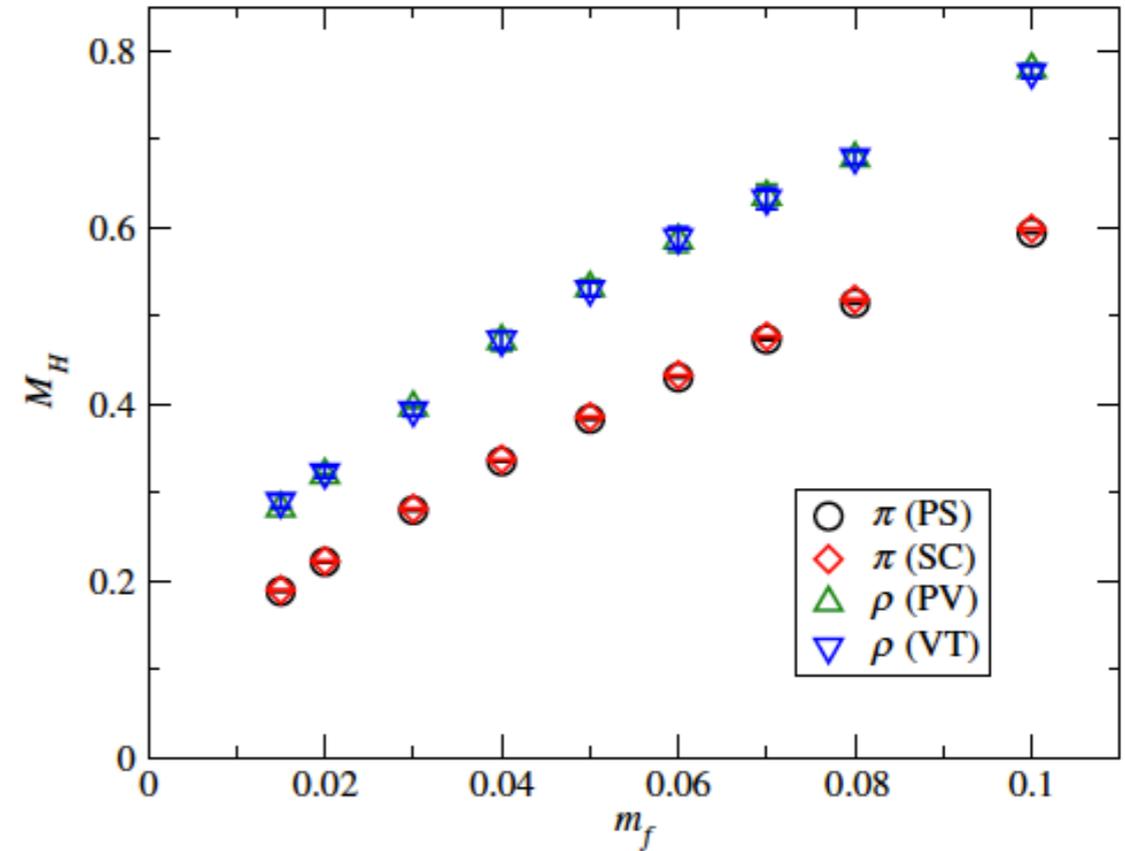
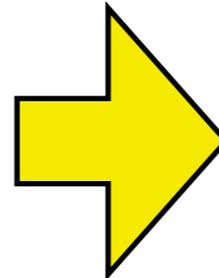
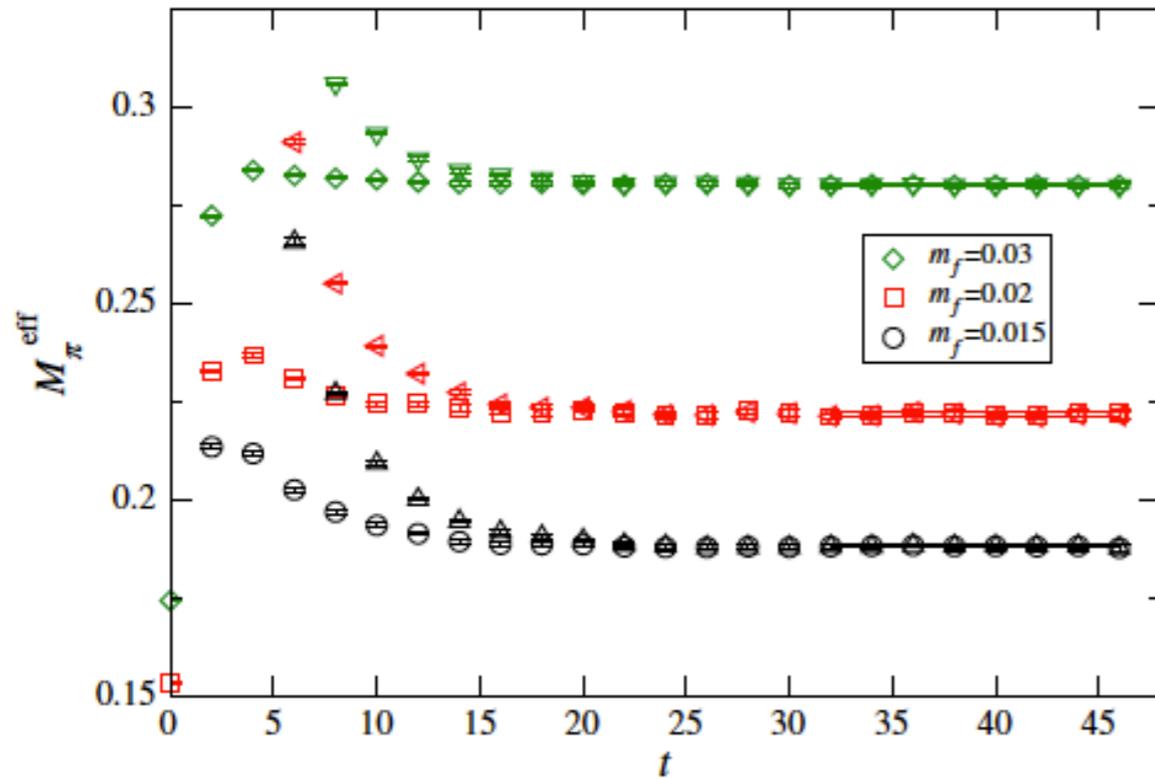
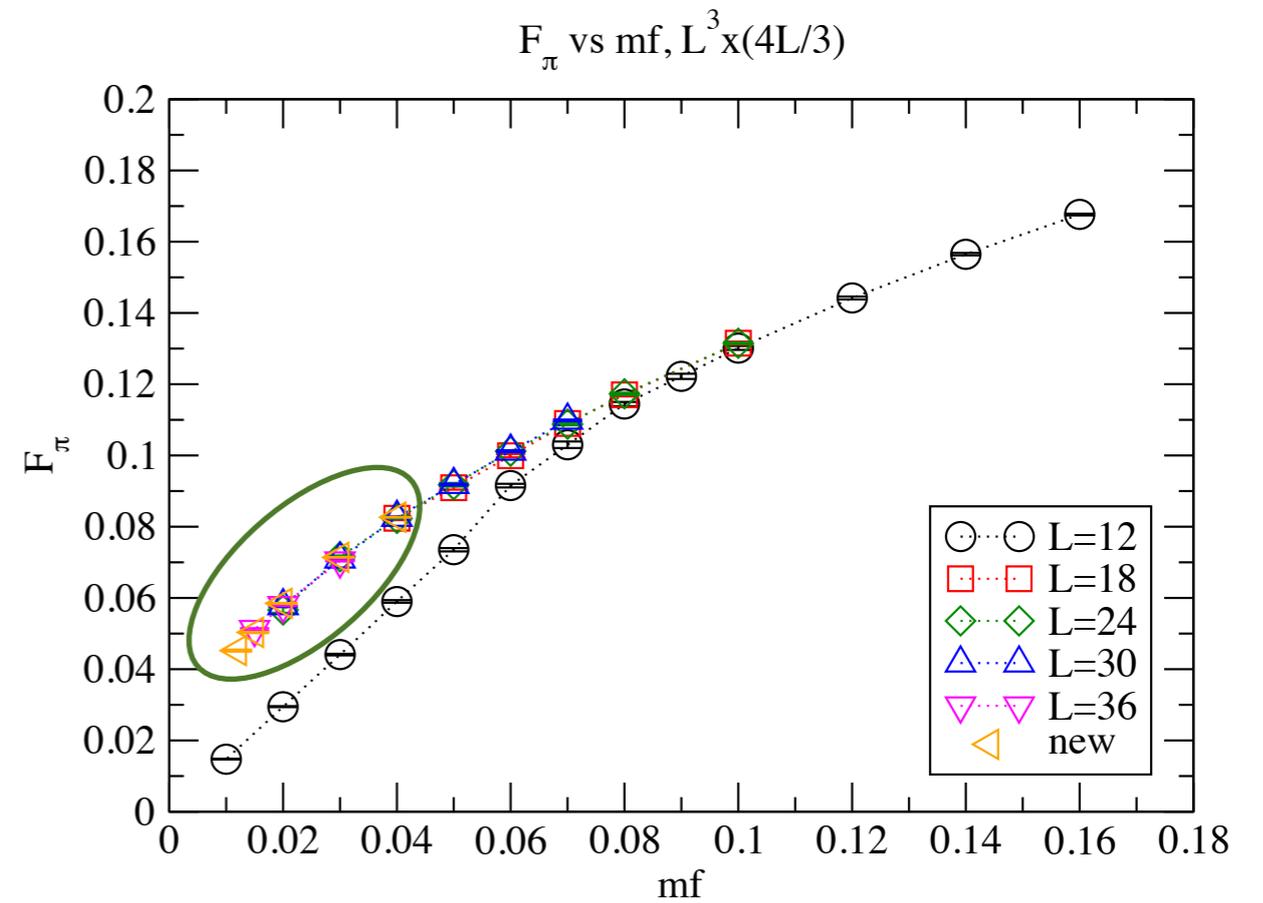
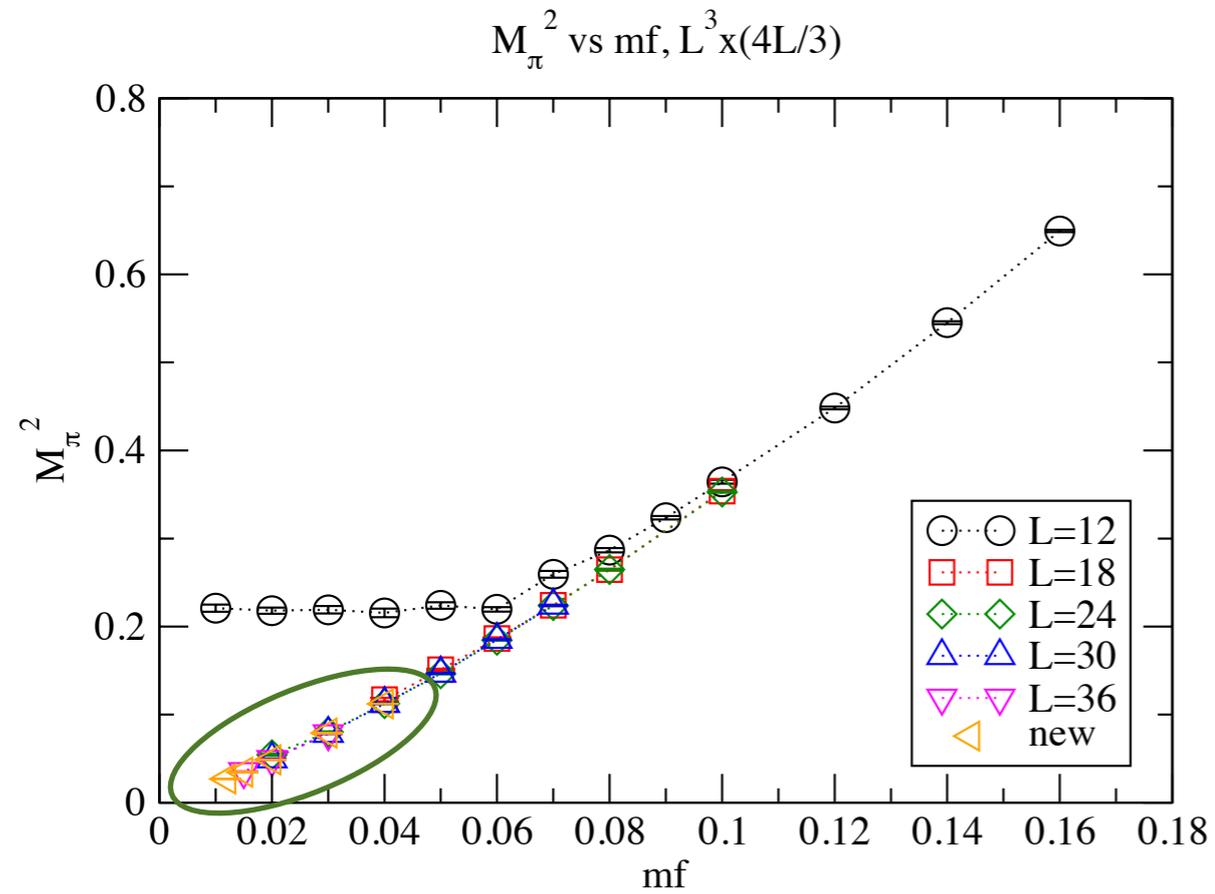


FIG. 2 (color online). Effective masses of PS meson,  $M_{\pi}^{\text{eff}}$ , at  $L = 36$ . Triangles and other symbols denote results from point sink correlators with random wall source and corner wall source, respectively. Fit results with error band obtained from random wall source correlator are also plotted by solid lines.

FIG. 4 (color online). Comparisons of  $M_{\pi}$  and  $M_{\text{SC}}$ , and of  $M_{\rho(\text{PV})}$  and  $M_{\rho(\text{VT})}$  as a function of  $m_f$  with largest volume data at each  $m_f$ .

data in the Nf=8 paper + updated data

$M_\pi^2, F_\pi$  vs.  $m_f$  on various  $L$

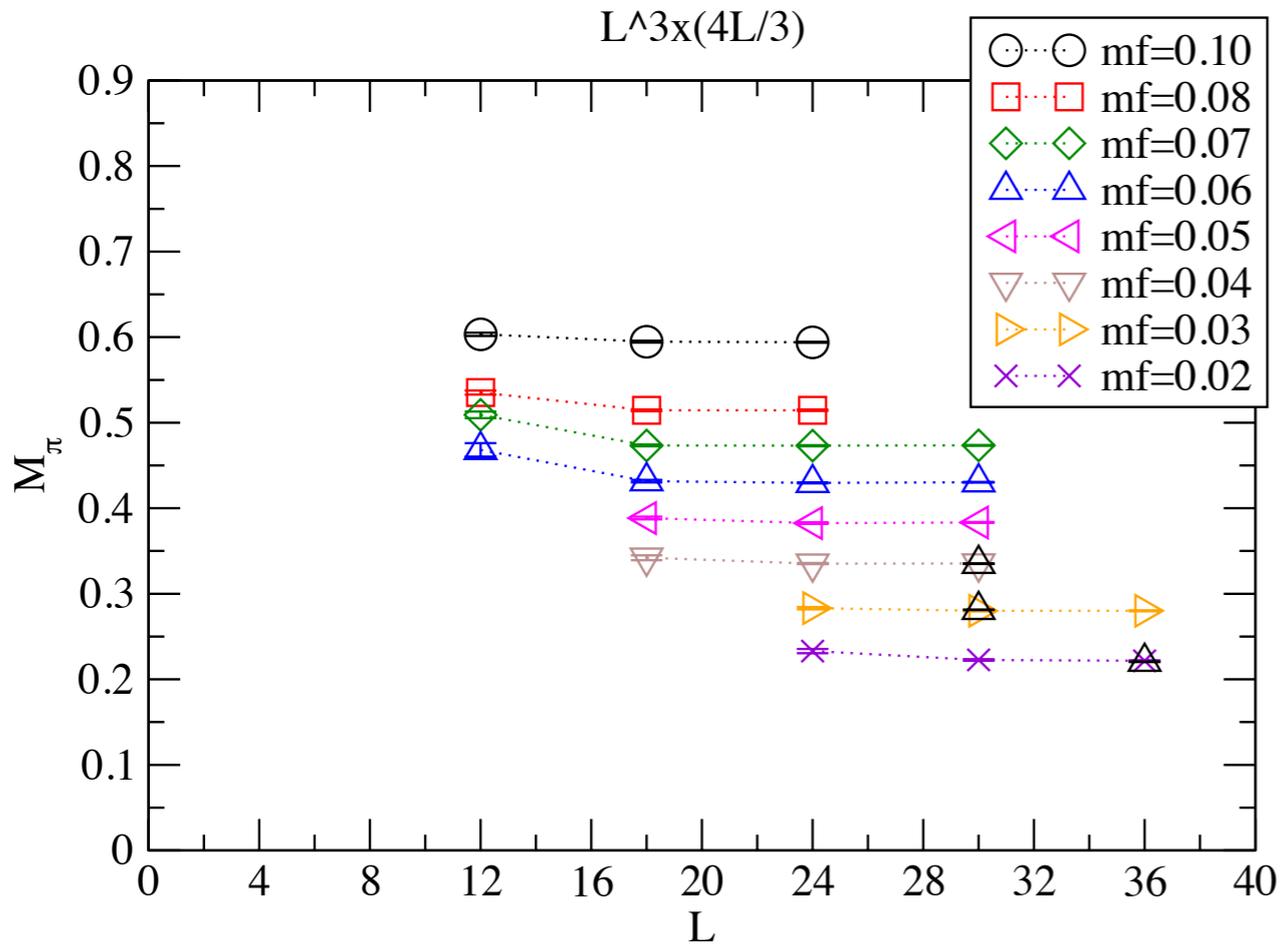


Preliminary

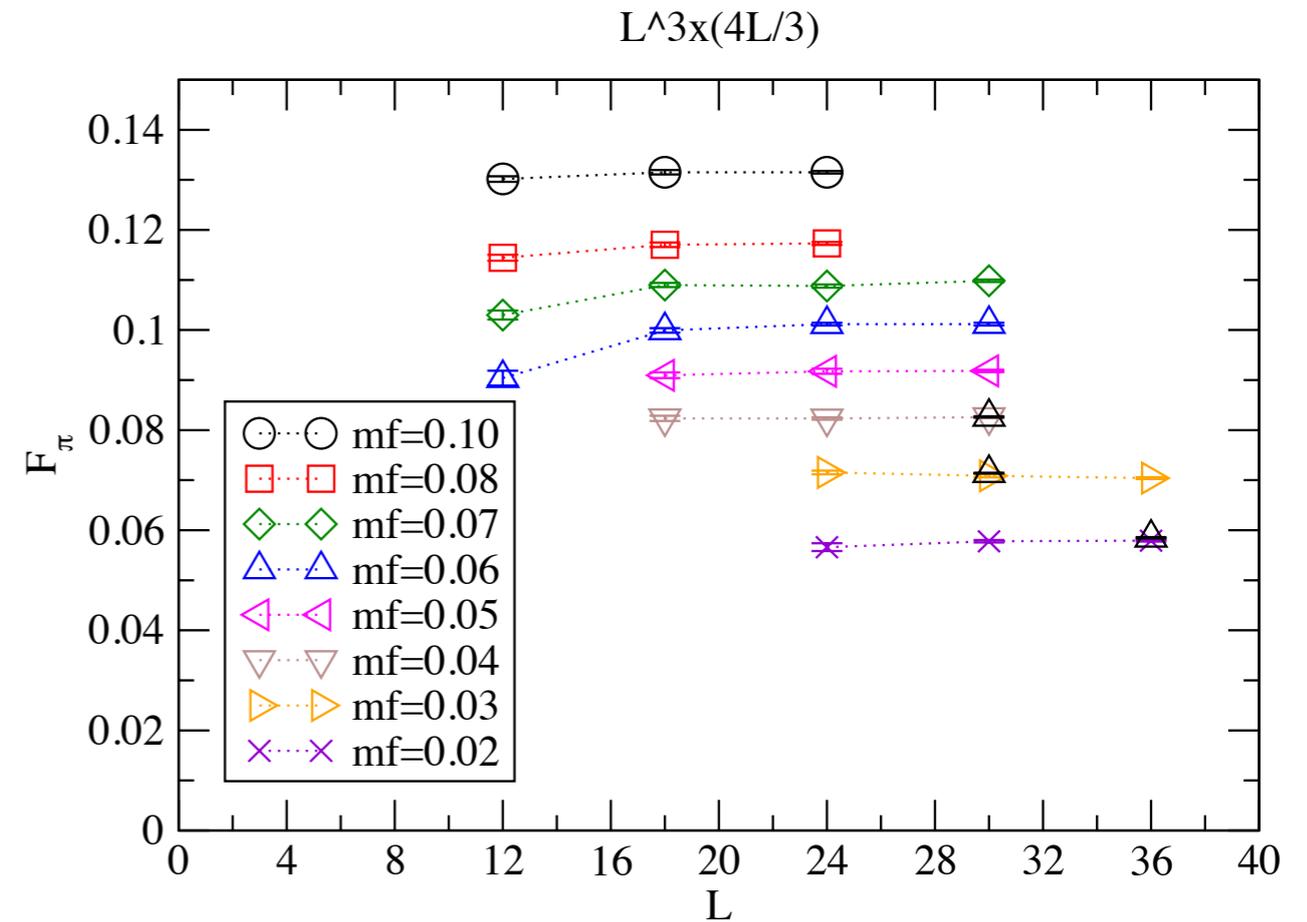
# Size dependence of $M_\pi$ and $F_\pi$ at $\beta=3.8$

$\triangle$ =updated

$M_\pi$  vs L at each mf



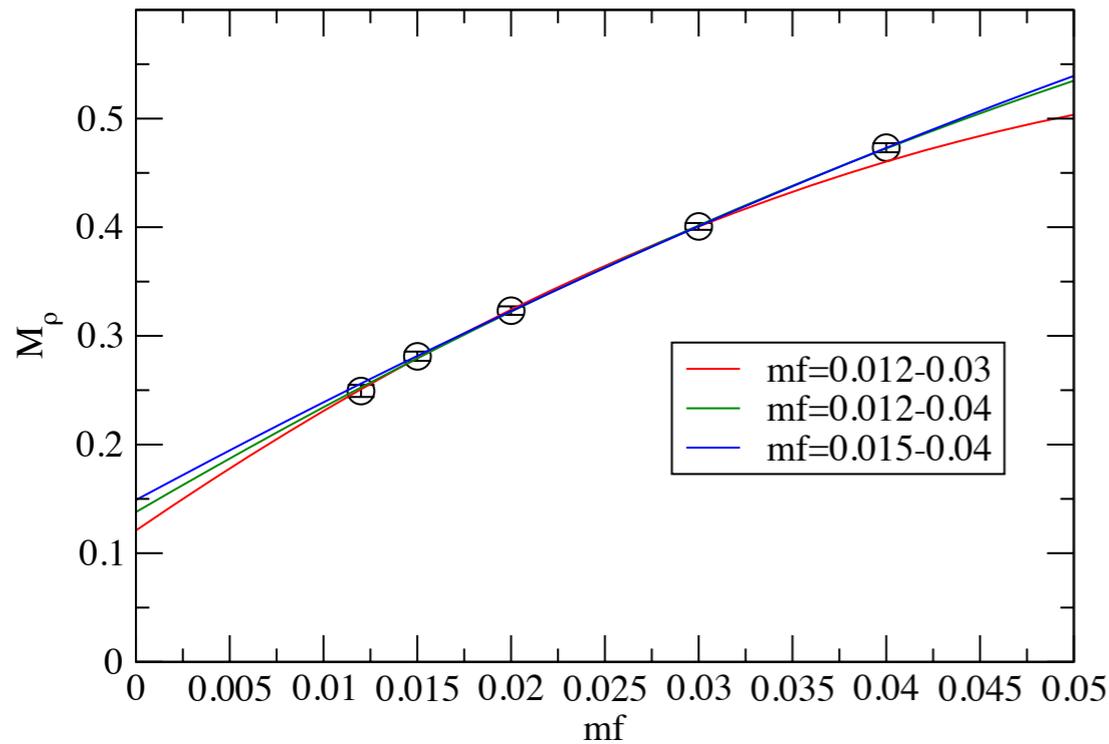
$F_\pi$  vs L at each mf



On the larger volume, there is not (or very tiny) size dependence.

# $M_\rho$ vs mf with quadratic-func. fit : $y=C_0+C_1*mf+C_2*mf^2$

$M_\rho$  vs mf, and quadratic fit



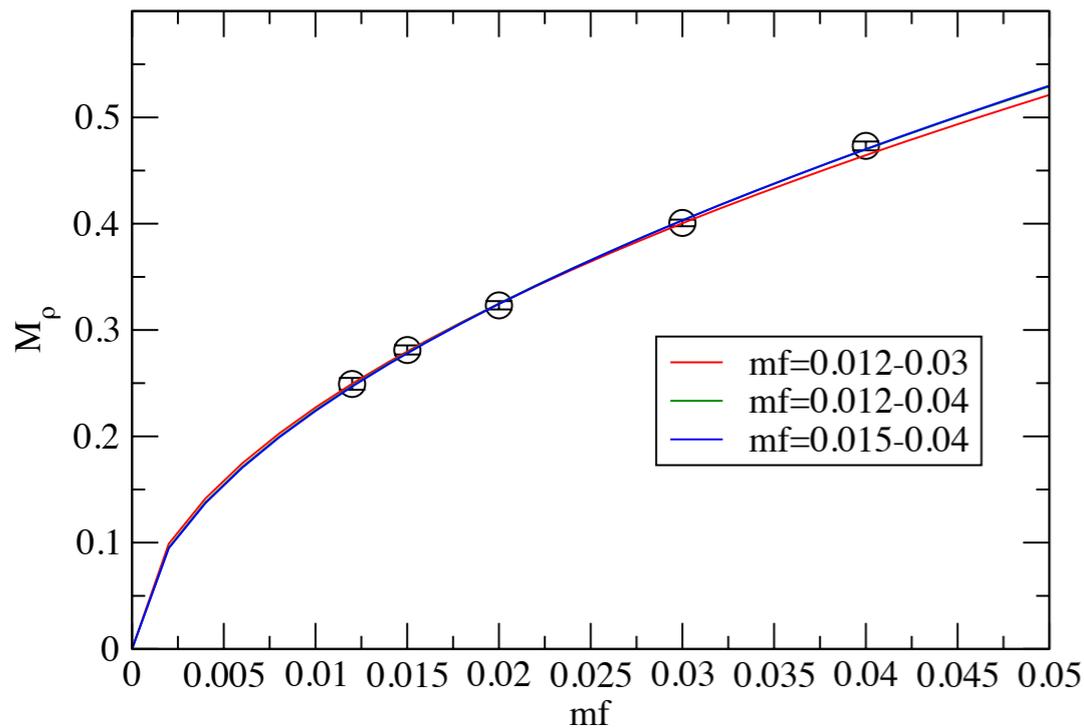
red :  $C_0=0.121(29)$ ,  $\chi^2/\text{dof}=0.27$  (dof=1)

green:  $C_0=0.138(15)$ ,  $\chi^2/\text{dof}=0.36$  (dof=2)

blue :  $C_0=0.149(20)$ ,  $\chi^2/\text{dof}=0.025$  (dof=1)

# $M_\rho$ vs mf with power-func. fit (hyperscaling) : $y=D_1*mf^\alpha$

$M_\rho$  vs mf, and power-func. fit w/o const.



red :  $\alpha=0.516(18)$ ,  $\chi^2/\text{dof}=0.12$  (dof=2)

green:  $\alpha=0.532(14)$ ,  $\chi^2/\text{dof}=0.62$  (dof=3)

blue :  $\alpha=0.535(16)$ ,  $\chi^2/\text{dof}=0.86$  (dof=2)

# Simultaneous fit of hyperscaling with mass corrections

$$\xi_H = C_0^H + C_1^H X + C_2^H L m_f^\alpha. \quad (\text{same method with } N_f=12)$$

Hyperscaling? in the middle region of  $m_f$  ( $m_f \geq 0.05$  and  $\xi_\pi (=M\pi L) \geq 8$ )

The mass corrections might be needed, as done in  $N_f=12$ ,  
from the lesson in SD analysis.

TABLE XI. Simultaneous FSHS fit with a correction term,  $\xi = C_0^H + C_1^H X + C_2^H L m_f^\alpha$  using several choices of  $\alpha$ . The fitted region is  $m_f \geq 0.05$  and  $\xi_\pi \geq 8$ .

$\alpha = 0.889(55)$	$C_0^H$	$C_1^H$	$C_2^H$
$\xi_\pi$	-0.005(25)	1.338(96)	1.494(37)
$\xi_F$	-0.0275(98)	0.4435(36)	—
$\xi_\rho$	0.53(16)	2.476(39)	—
$\gamma = 0.9130(76), \chi^2/\text{dof} = 1.73, \text{dof} = 33$			
$\alpha = 1$ fixed	$C_0^H$	$C_1^H$	$C_2^H$
$\xi_\pi$	-0.014(24)	1.61(10)	1.31(15)
$\xi_F$	-0.012(10)	0.484(30)	-0.068(44)
$\xi_\rho$	0.01(19)	2.60(17)	0.25(24)
$\gamma = 0.874(25), \chi^2/\text{dof} = 0.75, \text{dof} = 32$			
$\alpha = \frac{3-2\gamma}{1+\gamma}$ fixed	$C_0^H$	$C_1^H$	$C_2^H$
$\xi_\pi$	0.020(24)	1.52(39)	1.17(35)
$\xi_F$	-0.011(10)	0.572(34)	-0.158(52)
$\xi_\rho$	0.03(19)	2.91(30)	-0.15(36)
$\gamma = 0.775(56), \chi^2/\text{dof} = 0.93, \text{dof} = 32$			

$\Rightarrow$  good  $\chi^2/\text{dof}$ , but unclear which  $\alpha$  is better.